CHAPTER 12

Introduction to Compressible Flow

Case Study

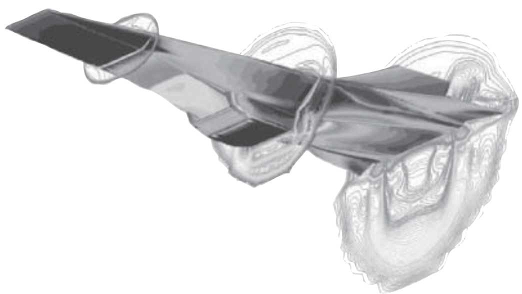
The X-43A/Hyper-X Airplane

Superman is faster than a speeding bullet. So how fast is that? It turns out that the highest speed of a bullet is about 1500 m/s, or about Mach 4.5 at sea level. Can humans keep up with Superman? If we're in orbit we can (What is the Mach number of the Space Shuttle in orbit?—it's a trick question!), because there's no drag—once we get up to speed, we can stay there—but flying at hypersonic speeds (i.e., above about $M = 5$) in the atmosphere requires tremendous engine thrust and an engine that can function at all at such speeds. In 2004, an air-breathing X-43A managed to fly at almost Mach 10, or about 11265.41 kph. The hypersonic scramjet engine in this airplane is actually integrated into the airframe, and the entire lower surface of the vehicle is shaped to make the engine work. The bulge on the underside in the figure is the engine. Unlike the turbojet engines used in many jet aircraft, which have fans and compressors as major components, the scramjet, amazingly, has no moving parts, so if you were to look inside it there wouldn't be much to see! Instead it uses geometry to develop a shock train that reduces the speed of the airflow from hypersonic to supersonic velocities. The scramjet, which is essentially a ramjet with supersonic combustion, doesn't need to slow the flow down to subsonic speeds. The compression ram on the undersurface of the aircraft slows the flow down from hypersonic to supersonic speed before it reaches the engine. It does this by causing a sequence of oblique shocks (which we discussed in this chapter) that successively slow the flow down and also increase the air density. As the supersonic, relatively high-density air passes through the engine, hydrogen fuel is injected and combusts, creating tremendous thrust at the exhaust. Once at hypersonic speed, the combustion process is self-sustaining.

One of the problems the engineers faced was how to start the engine. First, the airplane has to be accelerated above Mach 4 by conventional means (by a jet engine or rocket, or by piggy-backing another aircraft), and then the scramjet fuel can be started and ignited. This sounds simple enough, but the ignition process has been compared to "lighting a match in a hurricane"! The solution was to ignite using a mixture of pyrophoric silane (which auto-ignites when exposed to air) and hydrogen, then switch to pure hydrogen.

The X-43A/Hyper-X is experimental, but in the future, we may expect to see scramjets in military applications (aircraft and missiles), then possibly in commercial aircraft. Conceivably, you could live in New York, go to a meeting in Los Angeles, and be back in New York for dinner!

In this chapter, you will learn some of the basic ideas behind sub- and supersonic flow and why the designs of aircraft differ between the two regimes. You'll also learn about how shock waves form and why a supersonic nozzle looks so different from a subsonic one.



NASA Photo/NASA

The X-43A/Hyper-X at $M = 7$ (CFD image showing pressure contours).

In Chapter 2, we briefly discussed the two most important questions we must ask before analyzing a fluid flow: whether or not the flow is viscous, and whether or not the flow is compressible. We subsequently considered *incompressible, inviscid* flows (Chapter 6) and *incompressible, viscous* flows (Chapters 8 and 9). We are now ready to study flows that experience compressibility effects. Because this is an introductory text, our focus will be mainly on *one-dimensional compressible*, *inviscid* flows, although we will also review some important *compressible, viscous* flow phenomena.

We first need to establish what we mean by a "compressible" flow. This is a flow in which there are significant or noticeable changes in fluid density. Just as inviscid fluids do not actually exist, so incompressible fluids do not actually exist. For example, in this text we have treated water as an incompressible fluid, although in fact the density of seawater increases by 1 percent for each mile or so of depth. Hence, whether or not a given flow can be treated as incompressible is a judgment call: Liquid flows will almost always be considered incompressible (exceptions include phenomena such as the "water hammer" effect in pipes), but gas flows could easily be either incompressible or compressible. We will learn in Example 12.5 that for Mach numbers M less than about 0.3, the change in gas density due to the flow will be less than 3 percent; this change is small enough in most engineering applications for the following rule: *A gas flow can be considered incompressible when* $M < 0.3$.

The consequences of compressibility are not limited simply to density changes. Density changes mean that we can have significant compression or expansion work on a gas, so the thermodynamic state of the fluid will change, meaning that in general *all* properties—temperature, internal energy, entropy, and so on—can change. In particular, density changes create a mechanism (just as viscosity did) for exchange of energy between "mechanical" energies (kinetic, potential, and "pressure") and the thermal internal energy. For this reason, we begin with a review of the thermodynamics needed to study compressible flow.

After we cover the basic concepts of compressible flow, we will discuss one-dimensional compressible flow in more detail. We will look at what causes the fluid properties to vary in a one-dimensional compressible flow. Changes in the fluid properties can be caused by various phenomena, such as a varying flow area, a normal shock (which is a "violent" adiabatic process that causes the entropy to increase), friction on the walls of the flow passage, and heating or cooling. A real flow is likely to experience several of these phenomena simultaneously. Further, there may be two-dimensional flow effects, such as oblique shock and expansion waves. Although we will only introduce these subjects in this text, we hope it will provide you with a foundation for further study of this important topic.

12.1 *Review of Thermodynamics*

The pressure, density, temperature, and other properties of a substance may be related by an equation of state. Although many substances are complex in behavior, experience shows that most gases of engineering interest, at moderate pressure and temperature, are well represented by the ideal gas equation of state, (see References [1] or [2] for a review of the property relations for an ideal gas)

$$p = \rho RT \tag{12.1}$$

where R is a unique constant for each gas;[1] R is given by

$$R = \frac{R_u}{M_m}$$

where R_u is the universal gas constant, $R_u = 8314\ \mathrm{N \cdot m/(kgmole \cdot K)}$ and M_m is the molecular mass of the gas. Although the ideal gas equation is derived using a model that has the unrealistic assumptions that the gas molecules (a) take up zero volume (i.e., they are point masses) and (b) do not interact with one another, many real gases conform to Eq. 12.1, especially if the pressure is "low" enough and/or temperature "high" enough. For example, at room temperature, as long as the pressure is less than about 30 atm, Eq. 12.1 models the air density to better than 1 percent accuracy; similarly, Eq. 12.1 is accurate for air at 1 atm for temperatures that are greater than about $-130°\mathrm{C}(140\ \mathrm{K})$.

[1] For air, $R = 287\ \mathrm{N \cdot m/(kg \cdot K)}$.

The ideal gas has other features that are useful. In general, the *internal energy* of a simple substance may be expressed as a function of any two independent properties, e.g., $u = u(v,\ T)$, where $v \equiv 1/\rho$ is the *specific volume*. Then

$$du = \left(\frac{\partial u}{\partial T}\right)_v dT + \left(\frac{\partial u}{\partial v}\right)_T dv$$

The *specific heat at constant volume* is defined as $c_v \equiv (\partial u/\partial T)_v$, so that

$$du = c_v\, dT + \left(\frac{\partial u}{\partial v}\right)_T dv$$

In particular, for an ideal gas, the internal energy, u, is a function of temperature only, so $(\partial u/\partial v)_T = 0$, and

$$du = c_v\, dT \tag{12.2}$$

This means that internal energy and temperature changes may be related if c_v is known. Furthermore, since $u = u(T)$, then from Eq. 12.2, $c_v = c_v(T)$.

The *enthalpy* of any substance is defined as $h \equiv u + p/\rho$. For an ideal gas, $p = \rho RT$, and so $h = u + RT$. Since $u = u(T)$ for an ideal gas, h must also be a function of temperature alone.

We can obtain a relation between h and T by recalling once again that for a simple substance any property can be expressed as a function of any two other independent properties, e.g., $h = h(v,\ T)$ as we did for u, or $h = h(p, T)$. We choose the latter in order to develop a useful relation,

$$dh = \left(\frac{\partial h}{\partial T}\right)_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp$$

Since the *specific heat at constant pressure* is defined as $c_p \equiv (\partial h/\partial T)_p$,

$$dh = c_p\, dT + \left(\frac{\partial h}{\partial p}\right)_T dp$$

We have shown that for an ideal gas h is a function of T only. Consequently, $(\partial h/\partial T)_T = 0$ and

$$dh = c_p\, dT \tag{12.3}$$

Since h is a function of T alone, Eq. 12.3 requires that c_p be a function of T only for an ideal gas.

Although specific heats for an ideal gas are functions of temperature, their difference is a constant for each gas. To see this, from

$$h = u + RT$$

we can write

$$dh = du + R\,dT$$

Combining this with Eqs. 12.2 and 12.3, we can write

$$dh = c_p\, dT = du + R\, dT = c_v\, dT + R\, dT$$

Then

$$c_p - c_v = R \tag{12.4}$$

It may seem a bit odd that we have functions of temperature on the left of Eq. 12.4 but a constant on the right; it turns out that the specific heats of an ideal gas change with temperature at the same rate, so their *difference* is constant.

The *ratio of specific heats* is defined as

$$k \equiv \frac{c_p}{c_v} \tag{12.5}$$

Using the definition of k, we can solve Eq. 12.4 for either c_p or c_v in terms of k and R. Thus,

$$c_p = \frac{kR}{k-1} \tag{12.6a}$$

and

$$c_v = \frac{R}{k-1} \tag{12.6b}$$

Although the specific heats of an ideal gas may vary with temperature, for moderate temperature ranges they vary only slightly, and can be treated as constant, so

$$u_2 - u_1 = \int_{u_1}^{u_2} du = \int_{T_1}^{T_2} c_v \, dT = c_v(T_2 - T_1) \tag{12.7a}$$

$$h_2 - h_1 = \int_{h_1}^{h_2} dh = \int_{T_1}^{T_2} c_p \, dT = c_p(T_2 - T_1) \tag{12.7b}$$

Data for M_m, c_p, c_v, R, and k for common gases are given in Table A.6 of Appendix A.

We will find the property *entropy* to be extremely useful in analyzing compressible flows. State diagrams, particularly the temperature-entropy (Ts) diagram, are valuable aids in the physical interpretation of analytical results. Since we shall make extensive use of Ts diagrams in solving compressible flow problems, let us review briefly some useful relationships involving the property entropy.

Entropy is defined by the equation

$$\Delta S \equiv \int_{\text{rev}} \frac{\delta Q}{T} \quad \text{or} \quad dS = \left(\frac{\delta Q}{T}\right)_{\text{rev}} \tag{12.8}$$

where the subscript signifies *reversible*.

The inequality of Clausius, deduced from the second law, states that

$$\oint \frac{\delta Q}{T} \leq 0$$

As a consequence of the second law, we can write

$$dS \geq \frac{\delta Q}{T} \quad \text{or} \quad T\,dS \geq \delta Q \tag{12.9a}$$

For *reversible* processes, the equality holds, and

$$T\,ds = \frac{\delta Q}{m} (\text{reversible process}) \tag{12.9b}$$

The inequality holds for *irreversible* processes, and

$$T\,ds > \frac{\delta Q}{m} (\text{irreversible process}) \tag{12.9c}$$

For an *adiabatic* process, $\delta Q/m = 0$. Thus

$$ds = 0 \quad (\text{reversible adiabatic process}) \tag{12.9d}$$

and

$$ds > 0 \quad (\text{irreversible adiabatic process}) \tag{12.9e}$$

Thus a process that is *reversible and adiabatic* is also *isentropic*; the entropy remains constant during the process. Inequality 12.9e shows that entropy must *increase* for an adiabatic process that is irreversible. Equation 12.9 shows that any two of the restrictions—reversible, adiabatic, or isentropic—imply the third. For example, a process that is isentropic and reversible must also be adiabatic.

A useful relationship among properties (p, v, T, s, u) can be obtained by considering the first and second laws together. The result is the Gibbs, or $T\,ds$, equation

$$T\,ds = du + p\,dv \tag{12.10a}$$

This is a differential relationship among properties, valid for any process between any two equilibrium states. Although it is derived from the first and second laws, it is, in itself, a statement of neither.

An alternative form of Eq. 12.10a can be obtained by substituting

$$du = d(h - pv) = dh - p\,dv - v\,dp$$

to obtain

$$T\ ds = dh - v\ dp \tag{12.10b}$$

For an ideal gas, entropy change can be evaluated from the $T\ ds$ equations as

$$ds = \frac{du}{T} + \frac{p}{T}dv = c_v \frac{dT}{T} + R\frac{dv}{v}$$

$$ds = \frac{dh}{T} - \frac{v}{T}dp = c_p \frac{dT}{T} - R\frac{dp}{p}$$

For constant specific heats, these equations can be integrated to yield

$$s_2 - s_1 = c_v \ \ln\frac{T_2}{T_1} + R\ \ln\frac{v_2}{v_1} \tag{12.11a}$$

$$s_2 - s_1 = c_p \ \ln\frac{T_2}{T_1} - R\ \ln\frac{p_2}{p_1} \tag{12.11b}$$

and also

$$s_2 - s_1 = c_v \ \ln\frac{p_2}{p_1} + c_p\ \ln\frac{v_2}{v_1} \tag{12.11c}$$

Equation 12.11c can be obtained from either Eq. 12.11a or 12.11b using Eq. 12.4 and the ideal gas equation, Eq. 12.1, written in the form $pv = RT$, to eliminate T. Example 12.1 shows use of the above governing equations (the $T\ ds$ equations) to evaluate property changes during a process.

For an ideal gas with constant specific heats, we can use Eq. 12.11 to obtain relations valid for an isentropic process. From Eq. 12.11a

$$s_2 - s_1 = 0 = c_v \ \ln\frac{T_2}{T_1} + R\ \ln\frac{v_2}{v_1}$$

Then, using Eqs. 12.4 and 12.5,

$$\left(\frac{T_2}{T_1}\right)\left(\frac{v_2}{v_1}\right)^{R/c_v} = 0 \quad \text{or} \quad T_2 v_2^{k-1} = T_1 v_1^{k-1} = T v^{k-1} = \text{constant}$$

where states 1 and 2 are arbitrary states of the isentropic process. Using $v = 1/\rho$,

$$T v^{k-1} = \frac{T}{\rho^{k-1}} = \text{constant} \tag{12.12a}$$

We can apply a similar process to Eqs. 12.11b and 12.11c, respectively, and obtain the following useful relations:

$$T p^{1-k/k} = \text{constant} \tag{12.12b}$$

$$p v^k = \frac{p}{\rho^k} = \text{constant} \tag{12.12c}$$

Equation 12.12 is for an ideal gas undergoing an isentropic process.

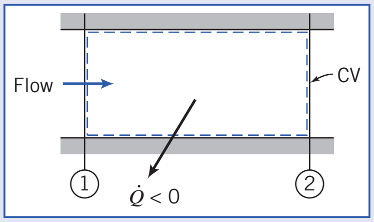
Qualitative information that is useful in drawing state diagrams can also be obtained from the $T\ ds$ equations. To complete our review of the thermodynamic fundamentals, we evaluate the slopes of lines of constant pressure and constant volume on the Ts diagram in Example 12.2.

Example 12.1 PROPERTY CHANGES IN COMPRESSIBLE DUCT FLOW

Air flows through a long duct of constant area at 0.15 kg/s. A short section of the duct is cooled by liquid nitrogen that surrounds the duct. The rate of heat loss in this section is 15.0 kJ/s from the air. The absolute pressure, temperature, and velocity entering the cooled section are 188 kPa, 440 K, and 210 m/s, respectively. At the outlet, the absolute pressure and temperature are 213 kPa and 351 K. Compute the duct cross-sectional area and the changes in enthalpy, internal energy, and entropy for this flow.

Given: Air flows steadily through a short section of constant-area duct that is cooled by liquid nitrogen.

$$T_1 = 440 \text{ K}$$
$$p_1 = 188 \text{ kPa (abs)}$$
$$V_1 = 210 \text{ m/s}$$



$$T_2 = 351 \text{ K}$$
$$p_2 = 213 \text{ kPa (abs)}$$

Find: (a) Duct area.
(b) Δh.
(c) Δu.
(d) Δs.

Solution: The duct area may be found from the continuity equation.

Governing equations:

$$\overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{\text{CV}} \rho d\forall + \int_{\text{CV}} \rho \vec{V} \cdot d\vec{A} = 0 \tag{4.12}$$

Assumptions:

1 Steady flow.
2 Uniform flow at each section.
3 Ideal gas with constant specific heats.

Then

$$(-\rho_1 V_1 A_1) + (\rho_2 V_2 A_2) = 0$$

or

$$\dot{m} = \rho_1 V_1 A = \rho_2 V_2 A$$

since $A = A_1 = A_2 =$ constant. Using the ideal gas relation, $p = \rho RT$, we find

$$\rho_1 = \frac{p_1}{RT_1} = 1.88 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg} \cdot \text{K}}{287 \text{ N} \cdot \text{m}} \times \frac{1}{440 \text{ K}} = 1.49 \text{ kg/m}^3$$

From continuity,

$$A = \frac{\dot{m}}{\rho_1 V_1} = 0.15 \frac{\text{kg}}{\text{s}} \times \frac{\text{m}^3}{1.49 \text{ kg}} \times \frac{\text{s}}{210 \text{ m}} = 4.79 \times 10^{-4} \text{ m}^2 \longleftarrow A$$

For an ideal gas, the change in enthalpy is

$$\Delta h = h_2 - h_1 = \int_{T_1}^{T_2} c_p \, dT = c_p (T_2 - T_1) \tag{12.7b}$$

$$\Delta h = 1.00 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} \times (351 - 440)\text{K} = -89.0 \text{ kJ/kg} \longleftarrow \Delta h$$

Also, the change in internal energy is

$$\Delta u = u_2 - u_1 = \int_{T_1}^{T_2} c_v \, dT = c_v (T_2 - T_1) \tag{12.7a}$$

$$\Delta u = 0.717 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} \times (351 - 440)\text{K} = -63.8 \text{ kJ/kg} \longleftarrow \Delta u$$

The entropy change may be obtained from Eq. 12.11b,

$$\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

$$= 1.00 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} \times \ln \left(\frac{351}{440} \right) - 0.287 \frac{\text{kJ}}{\text{kg} \cdot \text{K}} \times \ln \left(\frac{2.13 \times 10^5}{1.88 \times 10^5} \right)$$

$$\Delta s = -0.262 \text{ kJ/(kg} \cdot \text{K)} \longleftarrow \Delta s$$

We see that entropy may decrease for a nonadiabatic process in which the gas is cooled.

This problem illustrates the use of the governing equations for computing property changes of an ideal gas during a process.

Example 12.2 CONSTANT-PROPERTY LINES ON A *Ts* DIAGRAM

For an ideal gas, find the equations for lines of (a) constant volume and (b) constant pressure in the Ts plane.

Find: Equations for lines of (a) constant volume and (b) constant pressure in the Ts plane for an ideal gas.

Solution:

(a) We are interested in the relation between T and s with the volume v held constant. This suggests use of Eq. 12.11a,

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \overbrace{\cancel{\frac{v_2}{v_1}}}^{=0} \tag{12.8}$$

We relabel this equation so that state 1 is now reference state 0 and state 2 is an arbitrary state,

$$s - s_0 = c_v \ln \frac{T}{T_0} \quad \text{or} \quad T = T_0 e^{(s - s_0)/c_v} \tag{1}$$

Hence, we conclude that constant volume lines in the Ts plane are exponential.

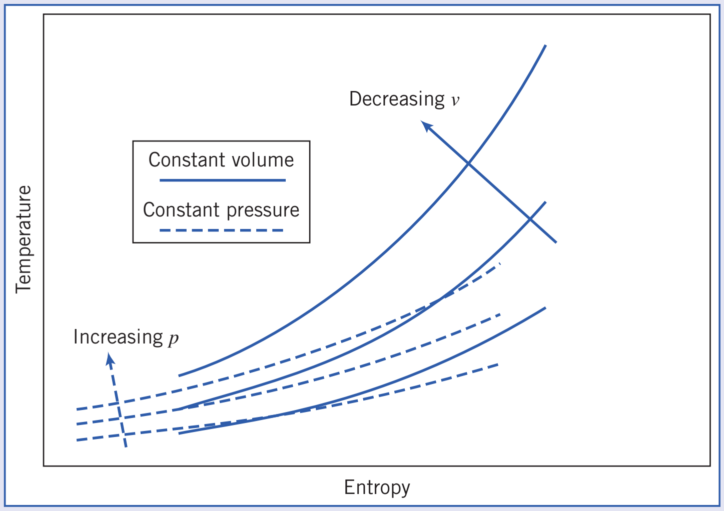
(b) We are interested in the relation between T and s with the pressure p held constant. This suggests use of Eq. 12.11b, and using a similar approach to case (a), we find

$$T = T_0 e^{(s - s_0)/c_p} \tag{2}$$

Hence, we conclude that constant pressure lines in the Ts plane are also exponential.

What about the slope of these curves? Because $c_p > c_v$ for all gases, we can see that the exponential, and therefore the slope, of the constant pressure curve, Eq. 2, is smaller than that for the constant volume curve, Eq. 1.

This is shown in the sketch below:



This problem illustrates use of governing equations to explore relations among properties.

12.2 *Propagation of Sound Waves*

Speed of Sound

A beginner to compressible flow studies might wonder what on earth sound has to do with the speeds present in a flow. We will see in this chapter and the next that the speed of sound, c, is an important marker in fluid mechanics: Flows with speeds less than the speed of sound are called *subsonic*; flows with speeds greater than the speed of sound are called *supersonic*; and we will learn that the behaviors of subsonic and supersonic flows are completely different. We have previously (in Chapters 2 and 7) defined the Mach number M of a flow, and it is so important for our studies that we redefine it here,

$$M \equiv \frac{V}{c} \tag{12.13}$$

where V is the speed of the fluid, or in some cases of the aircraft, so that $M < 1$ and $M > 1$ correspond to subsonic and supersonic flow, respectively. In addition, we mentioned in Section 12.1 that we'll demonstrate in Example 12.5 that for $M < 0.3$, we can generally assume incompressible flow. Hence, knowledge of the Mach number value is important in fluid mechanics.

An answer to the question posed at the beginning of this section is that the speed of sound is important in fluid mechanics because this is the speed at which "signals" can travel through the medium. Consider, for example, an object such as an aircraft in motion—the air ultimately has to move out of its way. In Newton's day, it was thought that this happened when the (invisible) air particles literally bounced off the front of the object, like so many balls bouncing off a wall; now we know that in most instances the air starts moving out of the way well before encountering the object; this will *not* be true when we have supersonic flow! How does the air "know" to move out of the way? It knows because as the object moves, it generates disturbances (infinitesimal pressure waves, which are sound waves) that emanate from the object in all directions. It is these waves that cumulatively "signal" the air and redirect it around the body as it approaches. These waves travel out at the speed of sound.

Sound is a pressure wave of very low pressure magnitude, for human hearing typically in the range of about 10^{-9} atm (the threshold of hearing) to about 10^{-3} atm (pain). Superimposed on the ambient atmospheric pressure, sound waves consist of extremely small pressure fluctuations. Because the range of human hearing covers about five or six orders of magnitude in pressure, typically we use a dimensionless logarithmic scale, the decibel level, to indicate sound intensity; 0 dB corresponds to the threshold of hearing, and if you listen to your MP3 player at full blast the sound will be at about 100 dB—about 10^{10} the intensity of the threshold of hearing!

Let us derive a method for computing the speed of sound in any medium (solid, liquid, or gas). As we do so, bear in mind that we are obtaining the speed of a "signal"—a pressure wave—and that the speed of the medium in which the wave travels is a completely different thing. For example, if you watch a soccer player kick the ball, a fraction of a second later you will hear the thud of contact as the sound, which is a pressure wave, travels from the field up to you in the stands, but no air particles traveled between you and the player. All of the air particles simply vibrated a bit.

Consider propagation of a sound wave of infinitesimal strength into an undisturbed medium, as shown in Fig. 12.1*a*. We are interested in relating the speed of wave propagation, c, to fluid property changes across the wave. If pressure and density in the undisturbed medium ahead of the wave are denoted by p and ρ, passage of the wave will cause them to undergo infinitesimal changes to become $p+dp$ and $\rho+d\rho$. Since the wave propagates into a stationary fluid, the velocity ahead of the wave, V_x, is zero. The magnitude of the velocity behind the wave, V_x+dV_x, then will be simply dV_x; in Fig. 12.1*a*, the direction of the motion behind the wave has been assumed to the left.[2]

The flow of Fig. 12.1*a* appears unsteady to a stationary observer, viewing the wave motion from a fixed point on the ground. However, the flow appears steady to an observer located *on* an inertial control volume moving with a segment of the wave, as shown in Fig. 12.1*b*. The velocity approaching the control volume is then c, and the velocity leaving is $c-dV_x$.

The basic equations may be applied to the differential control volume shown in Fig. 12.1*b* (we use V_x for the x component of velocity to avoid confusion with internal energy, u).

a. Continuity Equation

Governing equations:

$$\overbrace{\cancel{\frac{\partial}{\partial t}}\int_{\mathrm{CV}}\rho\, d\mathcal{V}}^{=0(1)}+\int_{\mathrm{CS}}\rho\vec{V}\cdot d\vec{A}=0 \tag{4.12}$$

Assumptions:

1 Steady flow.

2 Uniform flow at each section.

Then

$$(-\rho cA)+\{(\rho+d\rho)(c-dV_x)A\}=0 \tag{12.14a}$$

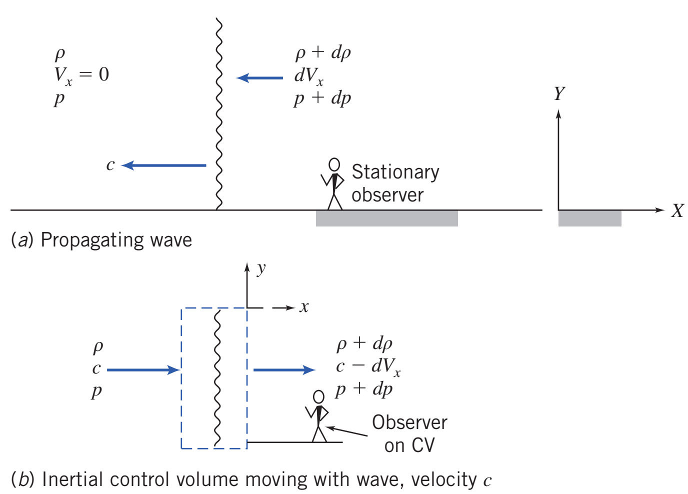


Fig. 12.1 Propagating sound wave showing control volume chosen for analysis.

[2] The same final result is obtained regardless of the direction initially assumed for motion behind the wave.

or

$$-\cancel{\rho c A}+\cancel{\rho c A}-\rho\, dV_x A+d\rho c A-d\rho\, d\overset{\approx 0}{\cancel{V_x}}A=0$$

or

$$dV_x=\frac{c}{\rho}d\rho \tag{12.14b}$$

b. Momentum Equation

Governing equation:

$$F_{S_x}+\overset{=0(3)}{\cancel{F_{B_x}}}=\overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}}\int_{\mathrm{CV}}V_x\rho\, d\forall+\int_{\mathrm{CS}}V_x\rho\vec{V}\cdot d\vec{A} \tag{4.18a}$$

Assumption:

3 $F_{B_x}=0$

The only surface forces acting in the x direction on the control volume of Fig. 12.1b are due to pressure. The upper and lower surfaces have zero friction because the areas are infinitesimal.

$$F_{S_x}=pA-(p+dp)A=-A\,dp$$

Substituting into the governing equation gives

$$-A\,dp=c(-\rho cA)+(c-dV_x)\{(\rho+d\rho)(c-dV_x)A\}$$

Using the continuity equation (Eq. 12.14a), this reduces to

$$-A\,dp=c(-\rho cA)+(c-dV_x)(\rho cA)=(-c+c-dV_x)(\rho cA)$$
$$-A\,dp=-\rho cA\,dV_x$$

or

$$dV_x=\frac{1}{\rho c}dp \tag{12.14c}$$

Combining Eqs. 12.14b and 12.14c, we obtain

$$dV_x=\frac{c}{\rho}d\rho=\frac{1}{\rho c}dp$$

from which

$$dp=c^2\,d\rho$$

or

$$c^2=\frac{dp}{d\rho} \tag{12.15}$$

We have derived an expression for the speed of sound in any medium in terms of thermodynamic quantities! Equation 12.15 indicates that the speed of sound depends on how the pressure and density of the medium are related. To obtain the speed of sound in a medium we could measure the time a sound wave takes to travel a prescribed distance, or instead we could apply a small pressure change dp to a sample, measure the corresponding density change $d\rho$, and evaluate c from Eq. 12.15. For example, an *incompressible* medium would have $d\rho=0$ for any dp, so $c\to\infty$. We can anticipate that solids and liquids whose densities are difficult to change will have relatively high c values, and gases whose densities are easy to change will have relatively low c values. There is only one problem with Eq. 12.15. For a simple substance, each property depends on any *two* independent properties. For a sound wave, by definition, we have an infinitesimal pressure change (i.e., it is *reversible*), and it occurs very quickly, so there is no time for any heat transfer to occur (i.e., it is *adiabatic*). Thus the sound wave propagates *isentropically*. Hence, if we express p as a function of density and entropy, $p=p(\rho,s)$, then

$$dp=\left(\frac{\partial p}{\partial \rho}\right)_s d\rho+\left(\frac{\partial p}{\partial s}\right)_\rho ds=\left(\frac{\partial p}{\partial \rho}\right)_s d\rho$$

so Eq. 12.15 becomes

$$c^2 = \frac{dp}{d\rho} = \left.\frac{\partial p}{\partial \rho}\right)_s$$

and

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right)_s} \tag{12.16}$$

We can now apply Eq. 12.16 to solids, liquids, and gases. For *solids* and *liquids*, data are usually available on the bulk modulus E_v, which is a measure of how a pressure change affects a relative density change,

$$E_v = \frac{dp}{d\rho/\rho} = \rho\frac{dp}{d\rho}$$

For these media

$$c = \sqrt{E_v/\rho} \tag{12.17}$$

For an *ideal gas*, the pressure and density in isentropic flow are related by

$$\frac{p}{\rho^k} = \text{constant} \tag{12.12c}$$

Taking logarithms and differentiating, we obtain

$$\frac{dp}{p} - k\frac{d\rho}{\rho} = 0$$

Therefore,

$$\left.\frac{\partial p}{\partial \rho}\right)_s = k\frac{p}{\rho}$$

But $p/\rho = RT$, so finally

$$c = \sqrt{kRT} \tag{12.18}$$

for an ideal gas. The speed of sound in air has been measured precisely by numerous investigators [3]. The results agree closely with the theoretical prediction of Eq. 12.18.

The important feature of sound propagation in an ideal gas, as shown by Eq. 12.18, is that the *speed of sound is a function of temperature only*. The variation in atmospheric temperature with altitude on a standard day was discussed in Chapter 3; the properties are summarized in Table A.3. Example 12.3 shows the use of Eqs. 12.17 and 12.18 in determining the speed of sound in different media.

Example 12.3 SPEED OF SOUND IN STEEL, WATER, SEAWATER, AND AIR

Find the speed of sound in (a) steel ($E_v \approx 200\,\text{GN/m}^2$), (b) water (at 20°C), (c) seawater (at 20°C), and (d) air at sea level on a standard day.

Find: Speed of sound in (a) steel ($E_v \approx 200\ \text{GN/m}^2$), (b) water (at 20°C), (c) seawater (at 20°C), and (d) air at sea level on a standard day.

Solution:

(a) For steel, a solid, we use Eq. 12.17, with ρ obtained from Table A.1(b),

$$c = \sqrt{E_v/\rho} = \sqrt{E_v/SG\rho_{H_2O}}$$

$$c = \sqrt{200 \times 10^9 \frac{N}{m^2} \times \frac{1}{7.83} \times \frac{1}{1000}\frac{m^3}{kg} \times \frac{kg \cdot m}{N \cdot s^2}} = 5050 \text{ m/s} \longleftarrow c_{steel}$$

(b) For water we also use Eq. 12.17, with data obtained from Table A.2,

$$c = \sqrt{E_v/\rho} = \sqrt{E_v/SG\rho_{H_2O}}$$

$$c = \sqrt{2.24 \times 10^9 \frac{N}{m^2} \times \frac{1}{0.998} \times \frac{1}{1000}\frac{m^3}{kg} \times \frac{kg \cdot m}{N \cdot s^2}} = 1500 \text{ m/s} \longleftarrow c_{water}$$

(c) For seawater we again use Eq. 12.17, with data obtained from Table A.2,

$$c = \sqrt{E_v/\rho} = \sqrt{E_v/SG\rho_{H_2O}}$$

$$c = \sqrt{2.42 \times 10^9 \frac{N}{m^2} \times \frac{1}{1.025} \times \frac{1}{1000}\frac{m^3}{kg} \times \frac{kg \cdot m}{N \cdot s^2}} = 1540 \text{ m/s} \longleftarrow c_{seawater}$$

(d) For air we use Eq. 12.18, with the sea level temperature obtained from Table A.3,

$$c = \sqrt{kRT}$$

$$c = \sqrt{1.4 \times 287 \frac{N \cdot m}{kg \cdot K} \times 288 \text{ K} \times \frac{kg \cdot m}{N \cdot s^2}} = 340 \text{ m/s} \longleftarrow c_{air(288\text{ K})}$$

This problem illustrates the relative magnitudes of the speed of sound in typical solids, liquids, and gases ($c_{solids} > c_{liquids} > c_{gases}$). Do not confuse the speed of sound with the *attenuation* of sound—the rate at which internal friction of the medium reduces the sound level—generally, solids and liquids attenuate sound much more rapidly than do gases.

Types of Flow—The Mach Cone

Flows for which $M < 1$ are *subsonic*, while those with $M > 1$ are *supersonic*. Flow fields that have both subsonic and supersonic regions are termed *transonic*. The transonic regime occurs for Mach numbers between about 0.9 and 1.2. Although most flows within our experience are subsonic, there are important practical cases where $M \geq 1$ occurs in a flow field. Perhaps the most obvious are supersonic aircraft and transonic flows in aircraft compressors and fans. Yet another flow regime, *hypersonic* flow ($M \gtrsim 5$), is of interest in missile and reentry-vehicle design. Some important qualitative differences between subsonic and supersonic flows can be deduced from the properties of a simple moving sound source.

Consider a point source of sound that emits a pulse every Δt seconds. Each pulse expands outward from its origination point at the speed of sound c, so at any instant t the pulse will be a sphere of radius ct centered at the pulse's origination point. We want to investigate what happens if the point source itself is moving. There are four possibilities, as shown in Fig. 12.2:

(a) $V = 0$. The point source is *stationary*. Figure 12.2*a* shows conditions after $3\Delta t$ seconds. The first pulse has expanded to a sphere of radius $c(3\Delta t)$, the second to a sphere of radius $c(2\Delta t)$, and the third to a sphere of radius $c(\Delta t)$; a new pulse is about to be emitted. The pulses constitute a set of ever-expanding concentric spheres.

(b) $0 < V < c$. The point source moves to the left at *subsonic* speed. Figure 12.2*b* shows conditions after $3\Delta t$ seconds. The source is shown at times $t = 0, \Delta t, 2\Delta t$, and $3\Delta t$. The first pulse has expanded to a sphere of radius $c(3\Delta t)$ *centered where the source was originally*, the second to a sphere of radius $c(2\Delta t)$ *centered where the source was at time* Δt, and the third to a sphere of radius $c(\Delta t)$ *centered where the source was at time* $2\Delta t$; a new pulse is about to be emitted. The pulses again constitute a set of ever-expanding spheres, except now they are not concentric. The pulses are all expanding at constant speed c. We make two important notes: First, we can see that an observer who is ahead of the source (or whom the source is approaching) will hear the pulses at a higher frequency rate than will an observer who is behind the source (this is the Doppler effect that occurs when a vehicle

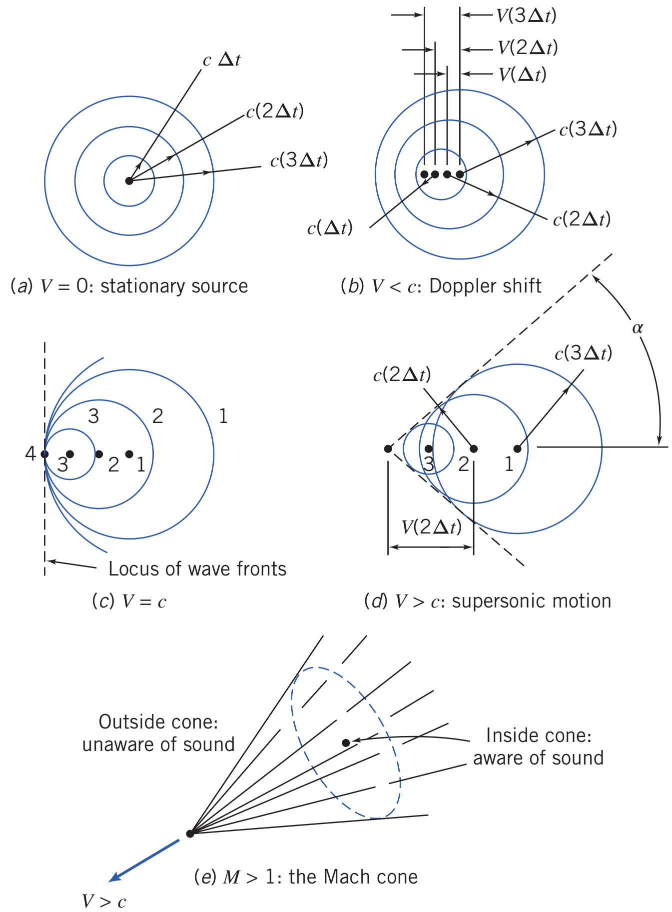


Fig. 12.2 Propagation of sound waves from a moving source: The Mach cone.

approaches and passes); second, an observer ahead of the source hears the source *before* the source itself reaches the observer.

(c) $V = c$. The point source moves to the left at *sonic* speed. Figure 12.2*c* shows conditions after $3\Delta t$ seconds. The source is shown at times $t = 0$ (point 1), Δt (point 2), $2\Delta t$ (point 3), and $3\Delta t$ (point 4). The first pulse has expanded to sphere 1 of radius $c(3\Delta t)$ *centered at point 1*, the second to sphere 2 of radius $c(2\Delta t)$ *centered at point 2*, and the third to sphere 3 of radius $c(\Delta t)$ *centered around the source at point 3*. We can see once more that the pulses constitute a set of ever-expanding spheres, except now they are tangent to one another on the left! The pulses are all expanding at constant speed c, but the source is also moving at speed c, with the result that the source and all its pulses are traveling together to the left. We again make two important notes: First, we can see that an observer who is ahead of the source will *not* hear the pulses before the source reaches the observer second, in theory, over time an unlimited number of pulses will accumulate at the front of the source, leading to a sound wave of unlimited amplitude. This was a source of concern to engineers trying to break the "sound barrier," which many people thought could not be broken—Chuck Yeager in a Bell X–1 was the first to do so in 1947.

(d) $V > c$. The point source moves to the left at *supersonic* speed. Figure 12.2*d* shows conditions after $3\Delta t$ seconds. By now it is clear how the spherical waves develop. We can see once more that the pulses constitute a set of ever-expanding spheres, except now the source is moving so fast it moves ahead of each sphere that it generates! For supersonic motion, the spheres generate what is called a *Mach cone* tangent to each sphere. The region inside the cone is called the *zone of action* and that outside the cone the *zone of silence*, for obvious reasons, as shown in Fig. 12.2*e*. From geometry, we see from Fig. 12.2*d* that

$$\sin\alpha = \frac{c}{V} = \frac{1}{M}$$

or

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) \tag{12.19}$$

Video: *Shock Waves due to a Projectile*

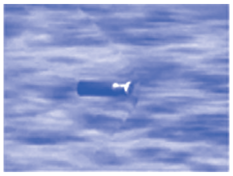
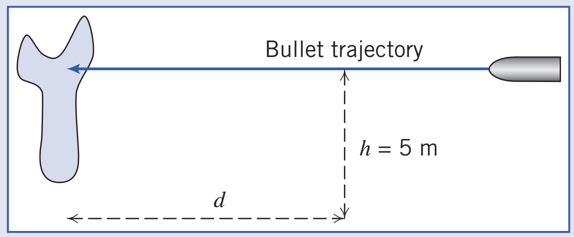


Figure 12.3 shows an image of an F/A–18 Hornet just as it accelerates to supersonic speed. The visible vapor pattern is due to the sudden increase in pressure as a shock wave washes over the aircraft (a shock wave leads to a sudden and large pressure increase). The (invisible) Mach cone emanates from the nose of the aircraft and passes through the periphery of the vapor disk. In Example 12.4, the properties of the Mach cone are used in analyzing a bullet trajectory.

Example 12.4 MACH CONE OF A BULLET

In tests of a protective material, we wish to photograph a bullet as it impacts a jacket made of the material. A camera is set up a perpendicular distance $h = 5$ m from the bullet trajectory. We wish to determine the perpendicular distance d from the target plane at which the camera must be placed such that the sound of the bullet will trigger the camera at the impact time. Note: The bullet speed is measured to be 550 m/s; the delay time of the camera is 0.005 s.



Find: Location of camera for capturing impact image.

Solution: The correct value of d is that for which the bullet hits the target 0.005 s before the Mach wave reaches the camera. We must first find the Mach number of the bullet; then we can find the Mach angle; finally, we can use basic trigonometry to find d.

Assuming sea level conditions, from Table A.3, we have $T = 288$ K. Hence Eq. 12.18 yields

$$c = \sqrt{kRT}$$

$$c = \sqrt{1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 288\ \text{K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}} = 340\ \text{m/s}$$

Then we can find the Mach number,

$$M = \frac{V}{c} = \frac{550\ \text{m/s}}{340\ \text{m/s}} = 1.62$$

From Eq. 12.19 we can next find the Mach angle,

$$\alpha = \sin^{-1}\left(\frac{1}{M}\right) = \sin^{-1}\left(\frac{1}{1.62}\right) = 38.2°$$

The distance x traveled by the bullet while the Mach wave reaches the camera is then

$$x = \frac{h}{\tan(\alpha)} = \frac{5\ \text{m}}{\tan(38.2°)} = 6.35\ \text{m}$$

Finally, we must add to this the time traveled by the bullet while the camera is operating, which is $0.005\ \text{s} \times 550\ \text{m/s}$,

$$d = 0.005\ \text{s} \times \frac{550\ \text{m}}{\text{s}} + 6.35\ \text{m} = 2.75\ \text{m} + 6.35\ \text{m}$$

$$d = 9.10\ \text{m} \longleftarrow d$$

Fig. 12.3 An F/A–18 Hornet as it breaks the sound barrier.

12.3 *Reference State: Local Isentropic Stagnation Properties*

In our study of compressible flow, we will discover that, in general, *all* properties $(p,\ T,\ \rho,\ u,\ s,\ V)$ may be changing as the flow proceeds. We need to obtain reference conditions that we can use to relate conditions in a flow from point to point. For any flow, a reference condition is obtained when the fluid is brought to rest either in reality or conceptually. We will call this the *stagnation condition*, and the property values $(p_0,\ T_0,\ \rho_0,\ u_0,\ h_0,\ s_0)$ at this state the *stagnation properties*. This process—of bringing the fluid to rest—is not as straightforward as it seems. For example, do we do so while there is friction, or while the fluid is being heated or cooled, or "violently," or in some other way? The simplest process to use is an isentropic process, in which there is no friction, no heat transfer, and no "violent" events. Hence, the properties we obtain will be the *local isentropic stagnation properties*. Why "local"? Because the actual flow can be any kind of flow, e.g., with friction, so it may or may not itself be isentropic. Hence, each point in the flow will have its own, or local, isentropic stagnation properties. This is illustrated in Fig. 12.4, showing a flow from some state ① to some new state ②. The local isentropic stagnation properties for each state, obtained by isentropically bringing the fluid to rest, are also shown. Hence, $s_{0_1} = s_1$ and $s_{0_2} = s_2$. The actual flow may or may not be isentropic. If it *is* isentropic, $s_1 = s_2 = s_{0_1} = s_{0_2}$, so the stagnation states are identical; if it is *not* isentropic, then $s_{0_1} \neq s_{0_2}$. We will see that changes in local isentropic stagnation properties will provide useful information about the flow.

We can obtain information on the reference isentropic stagnation state for *incompressible* flows by recalling the Bernoulli equation from Chapter 6

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{6.8}$$

valid for a steady, incompressible, frictionless flow along a streamline. Equation 6.8 is valid for an incompressible isentropic process because it is reversible (frictionless and steady) and adiabatic (we

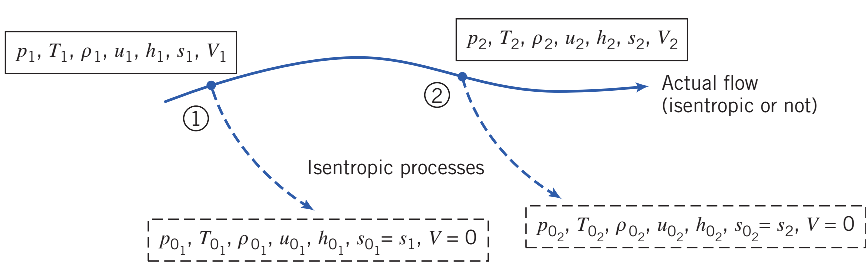


Fig. 12.4 Local isentropic stagnation properties.

did not include heat transfer considerations in its derivation). As we saw in Section 6.2, the Bernoulli equation leads to

$$p_0 = p + \frac{1}{2}\rho V^2 \tag{6.11}$$

The gravity term drops out because we assume the reference state is at the same elevation as the actual state, and in any event in external flows it is usually much smaller than the other terms. In Example 12.6, we compare isentropic stagnation conditions obtained assuming incompressibility (Eq. 6.11), and allowing for compressibility.

Local Isentropic Stagnation Properties for the Flow of an Ideal Gas

For a compressible flow we can derive the isentropic stagnation relations by applying the mass conservation and momentum equations to a differential control volume, and then integrating. For the process shown schematically in Fig. 12.4, we can depict the process from state ① to the corresponding stagnation state by imagining the control volume shown in Fig. 12.5. Consider first the continuity equation.

a. Continuity Equation

Governing equation:

$$\overbrace{\cancel{\frac{\partial}{\partial t}}}^{=0(1)}\int_{\mathrm{CV}} \rho\, d\forall + \int_{\mathrm{CS}} \rho \vec{V}\cdot d\vec{A} = 0 \tag{4.12}$$

Assumptions:

1 Steady flow.
2 Uniform flow at each section.

Then

$$(-\rho V_x A) + \{(\rho + d\rho)(V_x + dV_x)(A + dA)\} = 0$$

or

$$\rho V_x A = (\rho + d\rho)(V_x + dV_x)(A + dA) \tag{12.20a}$$

b. Momentum Equation

Governing equation:

$$F_{S_x} + \overbrace{\cancel{F_{B_x}}}^{=0(3)} = \overbrace{\cancel{\frac{\partial}{\partial t}}}^{=0(1)}\int_{\mathrm{CV}} V_x \rho\, d\forall + \int_{\mathrm{CS}} V_x \rho \vec{V}\cdot d\vec{A} \tag{4.18a}$$

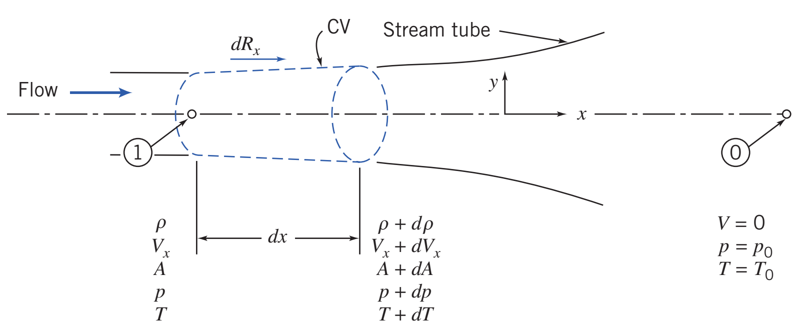


Fig. 12.5 Compressible flow in an infinitesimal stream tube.

Assumptions:

3 $F_{B_x} = 0$

4 Frictionless flow.

The surface forces acting on the infinitesimal control volume are

$$F_{S_x} = dR_x + pA - (p + dp)(A + dA)$$

The force dR_x is applied along the stream tube boundary, as shown in Fig. 12.5, where the average pressure is $p + dp/2$, and the area component in the x direction is dA. There is no friction. Thus,

$$F_{S_x} = \left(p + \frac{dp}{2}\right)dA + pA - (p + dp)(A + dA)$$

or

$$F_{S_x} = p\cancel{dA} + \overset{\approx 0}{\cancel{\frac{dp\,dA}{2}}} + \cancel{pA} - \cancel{pA} - dpA - p\cancel{dA} - \overset{\approx 0}{\cancel{dp\,dA}} = -dpA$$

Substituting this result into the momentum equation gives

$$-dp\ A = V_x\{-\rho V_x A\} + (V_x + dV_x)\{(\rho + d\rho)(V_x + dV_x)(A + dA)\}$$

which may be simplified using Eq. 12.20a to obtain

$$-dp\ A = (-V_x + V_x + dV_x)(\rho V_x A)$$

Finally,

$$dp = -\rho V_x dV_x = -\rho d\left(\frac{V_x^2}{2}\right)$$

or

$$\frac{dp}{\rho} + d\left(\frac{V_x^2}{2}\right) = 0 \tag{12.20b}$$

Equation 12.20b is a relation among properties during the deceleration process. (Note that for incompressible flow, it immediately leads to Eq. 6.11.) In developing this relation, we have specified a frictionless deceleration process. Before we can integrate between the initial state and final stagnation state, we must specify the relation that exists between pressure, p, and density, ρ, along the process path.

Since the deceleration process is isentropic, then p and ρ for an ideal gas are related by the expression

$$\frac{p}{\rho^k} = \text{constant} \tag{12.12c}$$

Our task now is to integrate Eq. 12.20b subject to this relation. Along the stagnation streamline, there is only a single component of velocity; V_x is the magnitude of the velocity. Hence we can drop the subscript in Eq. 12.20b.

From $p/\rho^k = \text{constant} = C$, we can write

$$p = C\rho^k \quad \text{and} \quad \rho = p^{1/k}C^{-1/k}$$

Then, from Eq. 12.20b,

$$-d\left(\frac{V^2}{2}\right) = \frac{dp}{\rho} = p^{-1/k}C^{1/k}dp$$

We can integrate this equation between the initial state and the corresponding stagnation state

$$-\int_V^0 d\left(\frac{V^2}{2}\right) = C^{1/k}\int_p^{p_0} p^{-1/k}dp$$

to obtain

$$\frac{V^2}{2} = C^{1/k}\frac{k}{k-1}\left[p^{(k-1)/k}\right]_p^{p_0} = C^{1/k}\frac{k}{k-1}\left[p_0^{(k-1)/k} - p^{(k-1)/k}\right]$$

$$\frac{V^2}{2} = C^{1/k}\frac{k}{k-1}p^{(k-1)/k}\left[\left(\frac{p_0}{p}\right)^{(k-1)/k} - 1\right]$$

Since $C^{1/k} = p^{1/k}/\rho$,

$$\frac{V^2}{2} = \frac{k}{k-1}\frac{p^{1/k}}{\rho}p^{(k-1)/k}\left[\left(\frac{p_0}{p}\right)^{(k-1)/k} - 1\right]$$

$$\frac{V^2}{2} = \frac{k}{k-1}\frac{p}{\rho}\left[\left(\frac{p_0}{p}\right)^{(k-1)/k} - 1\right]$$

Since we seek an expression for stagnation pressure, we can rewrite this equation as

$$\left(\frac{p_0}{p}\right)^{(k-1)/k} = 1 + \frac{k-1}{k}\frac{\rho}{p}\frac{V^2}{2}$$

and

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{k}\frac{\rho V^2}{2p}\right]^{k/(k-1)}$$

For an ideal gas, $p = \rho RT$, and hence

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}\frac{V^2}{kRT}\right]^{k/(k-1)}$$

Also, for an ideal gas the sonic speed is $c = \sqrt{kRT}$, and thus

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}\frac{V^2}{c^2}\right]^{k/(k-1)}$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{12.21a}$$

Equation 12.21a enables us to calculate the local isentropic stagnation pressure at any point in a flow field of an ideal gas, provided that we know the static pressure and Mach number at that point.

We can readily obtain expressions for other isentropic stagnation properties by applying the relation

$$\frac{p}{\rho^k} = \text{constant}$$

between end states of the process. Thus

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^k \quad \text{and} \quad \frac{\rho_0}{\rho} = \left(\frac{p_0}{p}\right)^{1/k}$$

For an ideal gas, then,

$$\frac{T_0}{T} = \frac{p_0}{p}\frac{\rho}{\rho_0} = \frac{p_0}{p}\left(\frac{p_0}{p}\right)^{-1/k} = \left(\frac{p_0}{p}\right)^{(k-1)/k}$$

Using Eq. 12.21a, we can summarize the equations for determining local isentropic stagnation properties of an ideal gas as

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{12.21a}$$

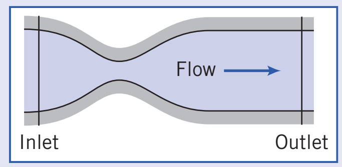
$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \tag{12.21b}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)} \tag{12.21c}$$

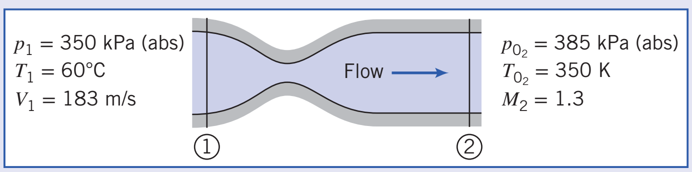
From Eq. 12.21, the ratio of each local isentropic stagnation property to the corresponding static property at any point in a flow field for an ideal gas can be found if the local Mach number is known. We will usually use Eq. 12.21 in lieu of the continuity and momentum equations for relating the properties at a state to that state's stagnation properties, but it is important to remember that we derived Eq. 12.21 using these equations *and* the isentropic relation for an ideal gas. Appendix D.1 lists flow functions for property ratios $T_0/T, p_0/p$, and ρ_0/ρ, in terms of M for isentropic flow of an ideal gas. A table of values, as well as a plot of these property ratios, is presented for air $(k = 1.4)$ for a limited range of Mach numbers. The associated *Excel* workbook, *Isentropic Relations*, available on the website, can be used to print a larger table of values for air and other ideal gases. The calculation procedure is illustrated in Example 12.5. The Mach number range for validity of the assumption of incompressible flow is investigated in Example 12.6.

Example 12.5 LOCAL ISENTROPIC STAGNATION CONDITIONS IN CHANNEL FLOW

Air flows steadily through the duct shown from 350 kPa (abs), 60°C, and 183 m/s at the inlet state to $M = 1.3$ at the outlet, where local isentropic stagnation conditions are known to be 385 kPa (abs) and 350 K. Compute the local isentropic stagnation pressure and temperature at the inlet and the static pressure and temperature at the duct outlet. Locate the inlet and outlet static state points on a Ts diagram and indicate the stagnation processes.



Given: Steady flow of air through a duct as shown in the sketch.



Find: (a) p_{0_1}.
(b) T_{0_1}.
(c) p_2.
(d) T_2.
(e) State points 1 and 2 on a Ts diagram; indicate the stagnation processes.

Solution: To evaluate local isentropic stagnation conditions at section 1, we must calculate the Mach number, $M_1 = V_1/c_1$. For an ideal gas, $c = \sqrt{kRT}$. Then

$$c_1 = \sqrt{kRT_1} = \left[1.4 \times 287 \frac{\mathrm{N \cdot m}}{\mathrm{kg \cdot K}} \times (273 + 60)\ \mathrm{K} \times \frac{\mathrm{kg \cdot m}}{\mathrm{N \cdot s^2}}\right]^{1/2} = 366\ \mathrm{m/s}$$

and

$$M_1 = \frac{V_1}{c_1} = \frac{183}{366} = 0.5$$

Local isentropic stagnation properties can be evaluated from Eqs. 12.21. Thus

$$p_{0_1} = p_1\left[1+\frac{k-1}{2}M_1^2\right]^{k(k-1)} = 350\text{ kPa}[1+0.2(0.5)^2]^{3.5} = 415\text{ kPa (abs)} \longleftarrow p_{0_1}$$

$$T_{0_1} = T_1\left[1+\frac{k-1}{2}M_1^2\right] = 333\text{ K}[1+0.2(0.5)^2] = 350\text{ K} \longleftarrow T_{0_1}$$

At section ②, Eq. 12.21 can be applied again. Thus from Eq. 12.21a,

$$p_2 = \frac{p_{0_2}}{\left[1+\frac{k-1}{2}M_2^2\right]^{k/(k-1)}} = \frac{385\text{ kPa}}{[1+0.2(1.3)^2]^{3.5}} = 139\text{ kPa (abs)} \longleftarrow p_2$$

From Eq. 12.21b,

$$T_2 = \frac{T_{0_2}}{1+\frac{k-1}{2}M_2^2} = \frac{350\text{ K}}{1+0.2(1.3)^2} = 262\text{ K} \longleftarrow T_2$$

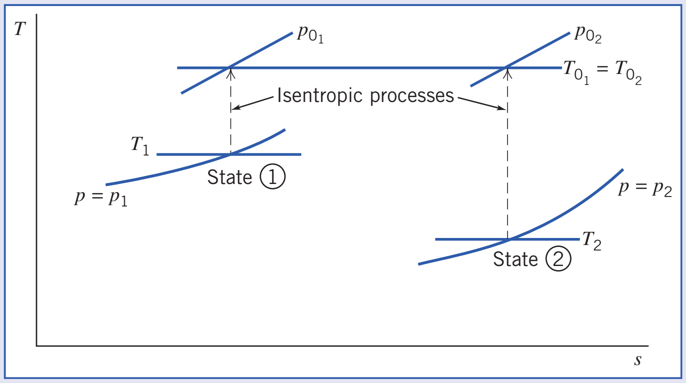
To locate states ① and ② in relation to one another and sketch the stagnation processes on the Ts diagram, we need to find the change in entropy $s_2 - s_1$. At each state we have p and T, so it is convenient to use Eq. 12.11b,

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

$$= 1.00\frac{\text{kJ}}{\text{kg}\cdot\text{K}} \times \ln\left(\frac{262}{333}\right) - 0.287\frac{\text{kJ}}{\text{kg}\cdot\text{K}} \times \ln\left(\frac{139}{350}\right)$$

$$s_2 - s_1 = 0.0252\text{ kJ/(kg}\cdot\text{K)}$$

Hence in this flow we have an increase in entropy. Perhaps there is irreversibility (e.g., friction), or heat is being added, or both. (We will see in Chapter 13 that the fact that $T_{0_1} = T_{0_2}$ for this particular flow means that actually we have an adiabatic flow.) We also found that $T_2 < T_1$ and that $p_2 < p_1$. We can now sketch the Ts diagram (and recall we saw in Example 12.2 that isobars (lines of constant pressure) in Ts space are exponential),



This problem illustrates use of the local isentropic stagnation properties (Eq. 12.21) to relate different points in a flow.

The *Excel* workbook *Isentropic Relations*, available on the website, can be used for computing property ratios from the Mach number M, as well as for computing M from property ratios.

Example 12.6 MACH-NUMBER LIMIT FOR INCOMPRESSIBLE FLOW

We have derived equations for p_0/p for both compressible and "incompressible" flows. By writing both equations in terms of Mach number, compare their behavior. Find the Mach number below which the two equations agree within engineering accuracy.

Given: The incompressible and compressible forms of the equations for stagnation pressure, p_0.

$$\text{Incompressible} \quad p_0 = p + \frac{1}{2}\rho V^2 \qquad (6.11)$$

$$\text{Compressible}\quad \frac{p_0}{p}=\left[1+\frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{12.21a}$$

Find: (a) Behavior of both equations as a function of Mach number.
(b) Mach number below which calculated values of p_0/p agree within engineering accuracy.

Solution: First, let us write Eq. 6.11 in terms of Mach number. Using the ideal gas equation of state and $c^2 = kRT$,

$$\frac{p_0}{p}=1+\frac{\rho V^2}{2p}=1+\frac{V^2}{2RT}=1+\frac{kV^2}{2kRT}=1+\frac{kV^2}{2c^2}$$

Thus,

$$\frac{p_0}{p}=1+\frac{k}{2}M^2 \tag{1}$$

for "incompressible" flow.

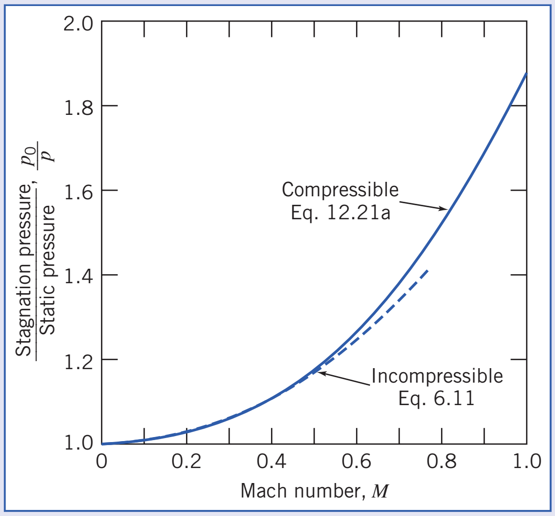
Equation 12.21a may be expanded using the binomial theorem,

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots,|x|<1$$

For Eq. 12.21a, $x=[(k-1)/2]M^2$, and $n=k/(k-1)$. Thus the series converges for $[(k-1)/2]/M^2<1$, and for compressible flow,

$$\begin{aligned}\frac{p_0}{p}&=1+\left(\frac{k}{k-1}\right)\left[\frac{k-1}{2}M^2\right]+\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\frac{1}{2!}\left[\frac{k-1}{2}M^2\right]^2\\&\quad+\left(\frac{k}{k-1}\right)\left(\frac{k}{k-1}-1\right)\left(\frac{k}{k-1}-2\right)\frac{1}{3!}\left[\frac{k-1}{2}M^2\right]^3+\cdots\\&=1+\frac{k}{2}M^2+\frac{k}{8}M^4+\frac{k(2-k)}{48}M^6+\cdots\\\frac{p_0}{p}&=1+\frac{k}{2}M^2\left[1+\frac{1}{4}M^2+\frac{(2-k)}{24}M^4+\cdots\right]\end{aligned} \tag{2}$$

In the limit, as $M\to 0$, the term in brackets in Eq. 2 approaches 1.0. Thus, for flow at low Mach number, the incompressible and compressible equations give the same result. The variation of p_0/p with Mach number is shown below. As Mach number is increased, the compressible equation gives a larger ratio, p_0/p.



Equations 1 and 2 may be compared quantitatively most simply by writing

$$\frac{p_0}{p}-1=\frac{k}{2}M^2\text{("incompressible")}$$

$$\frac{p_0}{p}-1=\frac{k}{2}M^2\left[1+\frac{1}{4}M^2+\frac{(2-k)}{24}M^4+\cdots\right]\text{(compressible)}$$

12.4 *Critical Conditions*

Stagnation conditions are extremely useful as reference conditions for thermodynamic properties; this is not true for velocity, since by definition $V = 0$ at stagnation. A useful reference value for velocity is the *critical speed*—the speed V we attain when a flow is either accelerated or decelerated (actually or conceptually) isentropically until we reach $M = 1$. Even if there is no point in a given flow field where the Mach number is equal to unity, such a hypothetical condition still is useful as a reference condition.

Using asterisks to denote conditions at $M = 1$, then by definition

$$V^* \equiv c^*$$

At critical conditions, Eq. 12.21 for isentropic stagnation properties become

$$\frac{p_0}{p*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \tag{12.22a}$$

$$\frac{T_0}{T^*} = \frac{k+1}{2} \tag{12.22b}$$

$$\frac{\rho_0}{\rho^*} = \left[\frac{k+1}{2}\right]^{1/(k-1)} \tag{12.22c}$$

The critical speed may be written in terms of either critical temperature, T^*, or isentropic stagnation temperature, T_0.

For an ideal gas, $c^* = \sqrt{kRT^*}$, and thus $V^* = \sqrt{kRT^*}$. Since, from Eq. 12.22b,

$$T^* = \frac{2}{k+1}T_0$$

we have

$$V^* = c^* = \sqrt{\frac{2k}{k+1}RT_0} \tag{12.23}$$

We shall use both stagnation conditions and critical conditions as reference conditions later in this chapter when we consider a variety of compressible flows.

12.5 *Basic Equations for One-Dimensional Compressible Flow*

Our first task is to develop general equations for a one-dimensional flow that express the basic laws from Chapter 4: *mass conservation* (continuity), *momentum*, the *first law of thermodynamics*, the *second law of thermodynamics*, and an *equation of state*. To do so, we will use the fixed control volume shown in Fig. 12.6. We initially assume that the flow is affected by *all* of the phenomena mentioned above (i.e., area change, friction, and heat transfer—even the normal shock will be described by this approach). Then, for each individual phenomenon, we will simplify the equations to obtain useful results.

As shown in Fig. 12.6, the properties at sections 1 and 2 are labeled with corresponding subscripts. R_x is the x component of surface force from friction and pressure on the sides of the channel. There will also be surface forces from pressures at surfaces 1 and 2. Note that the x component of body force is zero, so it is not shown. $\dot{Q}$ is the heat transfer.

Continuity Equation

Basic equation:

$$\overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}\int_{\text{CV}} \rho \, d\forall} + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} = 0 \tag{4.12}$$

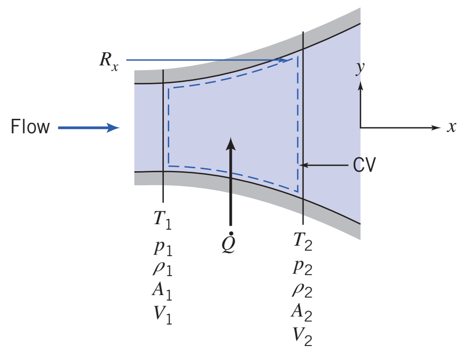


Fig. 12.6 Control volume for analysis of a general one-dimensional flow.

Assumptions:

1 Steady flow.
2 One-dimensional flow.

Then

$$(-\rho_1 V_1 A_1) + (\rho_2 V_2 A_2) = 0$$

or

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \qquad (12.24a)$$

Momentum Equation

Basic equation:

$$F_{S_x} + \overset{=0(3)}{\cancel{F_{B_x}}} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{\text{CV}} V_x \rho \, d\forall + \int_{\text{CS}} V_x \rho \vec{V} \cdot d\vec{A} \qquad (4.18a)$$

Assumption:

3 $F_{B_x} = 0$

The surface force is caused by pressure forces at surfaces ① and ②, and by the friction and distributed pressure force, R_x, along the channel walls. Substituting gives

$$R_x + p_1 A_1 - p_2 A_2 = V_1(-\rho_1 V_1 A_1) + V_2(\rho_2 V_2 A_2)$$

Using continuity, we obtain

$$R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1 \qquad (12.24b)$$

First Law of Thermodynamics

Basic equation:

$$\dot{Q} - \cancel{\dot{W}_s} - \cancel{\dot{W}}_{\text{shear}} - \cancel{\dot{W}}_{\text{other}} = \cancel{\frac{\partial}{\partial t}} \int_{\text{CV}} e \rho \, d\forall + \int_{\text{CS}} (e + pv) \rho \vec{V} \cdot d\vec{A} \qquad (4.56)$$

where

$$e = u + \frac{V^2}{2} + \overset{\simeq 0(6)}{\cancel{gz}}$$

Assumptions:

4 $\dot{W}_s = 0$.
5 $\dot{W}_{\text{shear}} = \dot{W}_{\text{other}} = 0$.
6 Effects of gravity are negligible.

Note that even if we have friction, there is no friction *work* at the walls because with friction the velocity at the walls must be zero from the no-slip condition. Under these assumptions, the first law reduces to

$$\dot{Q} = \left(u_1 + p_1 v_1 + \frac{V_1^2}{2}\right)(-\rho_1 V_1 A_1) + \left(u_2 + p_2 v_2 + \frac{V_2^2}{2}\right)(\rho_2 V_2 A_2)$$

(Remember that v here represents the specific volume.) This can be simplified by using $h \equiv u + pv$, and continuity (Eq. 12.24a),

$$\dot{Q} = \dot{m}\left[\left(h_2 + \frac{V_2^2}{2}\right) - \left(h_1 + \frac{V_1^2}{2}\right)\right]$$

We can write the heat transfer on a per unit mass rather than per unit time basis:

$$\frac{\delta Q}{dm} = \frac{1}{\dot{m}}\dot{Q}$$

so

$$\frac{\delta Q}{dm} + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{12.24c}$$

Equation 12.24c expresses the fact that heat transfer changes the total energy (the sum of thermal energy h and kinetic energy $V^2/2$) of the flowing fluid. This combination, $h + V^2/2$, occurs often in compressible flow and is called the *stagnation enthalpy*, h_0. This is the enthalpy obtained if a flow is brought adiabatically to rest.

Hence, Eq. 12.24c can also be written

$$\frac{\delta Q}{dm} = h_{0_2} - h_{0_1}$$

We see that heat transfer causes the stagnation enthalpy, and hence, stagnation temperature, T_0, to change.

Second Law of Thermodynamics

Basic equation:

$$\overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}\int_{\text{CV}} s\,\rho\,d\forall} + \int_{\text{CS}} s\,\rho\vec{V}\cdot d\vec{A} \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA \tag{4.58}$$

or

$$s_1(-\rho_1 V_1 A_1) + s_2(\rho_2 V_2 A_2) \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA$$

and, again using continuity,

$$\dot{m}(s_2 - s_1) \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA \tag{12.24d}$$

Equation of State

Equations of state are relations among intensive thermodynamic properties. These relations may be available as tabulated data or charts, or as algebraic equations. In general, regardless of the format of the data, as we discussed in earlier in this chapter, for a simple substance, any property can be expressed as a function of any two other independent properties. For example, we could write $h = h(s, p)$, or $\rho = \rho(s, p)$, and so on.

We will primarily be concerned with ideal gases with constant specific heats, and for these we can write Eqs. 12.1 and 12.7b (renumbered for convenient use in this chapter),

$$p = \rho RT \tag{12.24e}$$

and

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1) \tag{12.24f}$$

For ideal gases with constant specific heats, the change in entropy, $\Delta s = s_2 - s_1$, for any process can be computed from any of Eq. 12.11. For example, Eq. 12.11b (renumbered here for convenience) is

$$\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{12.24g}$$

We now have a basic set of equations for analyzing one-dimensional compressible flows of an ideal gas with constant specific heats:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \tag{12.24a}$$

$$R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1 \tag{12.24b}$$

$$\frac{\delta Q}{dm} + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{12.24c}$$

$$\dot{m}(s_2 - s_1) \geq \int_{\text{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) dA \tag{12.24d}$$

$$p = \rho RT \tag{12.24e}$$

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1) \tag{12.24f}$$

$$\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{12.24g}$$

Note that Eq. 12.24e applies only if we have an ideal gas; Equations 12.24f and 12.24g apply only if we have an ideal gas with constant specific heats. Our task is now to simplify this set of equations for each of the phenomena that can affect the flow:

- Flow with varying area.
- Normal shock.
- Flow in a channel with friction.
- Flow in a channel with heating or cooling.

12.6 *Isentropic Flow of an Ideal Gas: Area Variation*

The first phenomenon is one in which the flow is changed only by area variation—there is no heat transfer, friction, or shocks. The absence of heat transfer, friction, and shocks means that the flow will be reversible and adiabatic, so Eq. 12.24d becomes

$$\dot{m}(s_2 - s_1) = \int_{\text{CS}} \frac{1}{T} \left(\frac{\dot{Q}}{A} \right) dA = 0$$

or

$$\Delta s = s_2 - s_1 = 0$$

so such a flow is *isentropic*. This means that Eq. 12.24g leads to the result we saw previously,

$$T_1 p_1^{(1-k)/k} = T_2 p_2^{(1-k)/k} = T p^{(1-k)/k} = \text{constant} \tag{12.12b}$$

or its equivalent, which can be obtained by using the ideal gas equation of state in Eq. 12.12b to eliminate temperature,

$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} = \frac{p}{\rho^k} = \text{constant} \tag{12.12c}$$

Hence, the basic set of equations (Eq. 12.24) becomes:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \tag{12.25a}$$

$$R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1 \tag{12.25b}$$

$$h_{0_1} = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{0_2} = h_0 \tag{12.25c}$$

$$s_2 = s_1 = s \tag{12.25d}$$

$$p = \rho R T \tag{12.25e}$$

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1) \tag{12.25f}$$

$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} = \frac{p}{\rho^k} = \text{constant} \tag{12.25g}$$

Note that Eqs. 12.25c, 12.25d, and 12.25f provide insight into how this process appears on an *hs* diagram and on a *Ts* diagram. From Eq. 12.25c, the total energy, or stagnation enthalpy h_0, of the fluid is constant; the enthalpy and kinetic energy may vary along the flow, but their sum is constant. This means that if the fluid accelerates, its temperature must decrease, and vice versa. Equation 12.25d indicates that the entropy remains constant. These results are shown for a typical process in Fig. 12.7.

Equation 12.25f indicates that the temperature and enthalpy are linearly related; hence, processes plotted on a *Ts* diagram will look very similar to that shown in Fig. 12.7 except for the vertical scale.

Equation 12.25 *could* be used to analyze isentropic flow in a channel of varying area. For example, if we know conditions at section ① (i.e., p_1, ρ_1, T_1, s_1, h_1, V_1, and A_1) we could use these equations to find conditions at some new section ② where the area is A_2: We would have seven equations and seven unknowns (p_2, ρ_2, T_2, s_2, h_2, V_2, and R_x). We stress *could*, because in practice this procedure is unwieldy—we have a set of seven *nonlinear coupled algebraic* equations to solve. Instead we can take advantage of the results we obtained for isentropic flows and develop property relations in terms of the local Mach number, the stagnation conditions, and critical conditions.

Before proceeding with this approach, we can gain insight into the isentropic process by reviewing the results we obtained previously when we analyzed a differential control volume (Fig. 12.5). The momentum equation for this was

$$\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) = 0 \tag{12.20b}$$

Then

$$dp = -\rho V\, dV$$

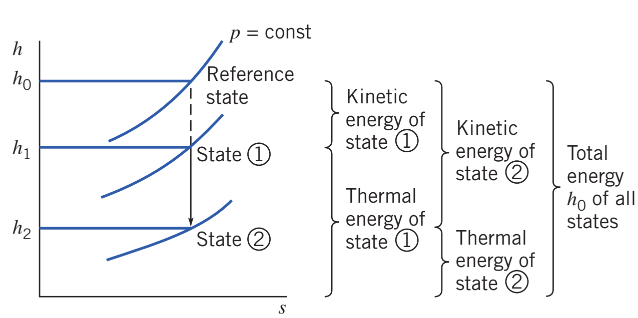


Fig. 12.7 Isentropic flow in the *hs* plane.

Dividing by ρV^2, we obtain

$$\frac{dp}{\rho V^2} = -\frac{dV}{V} \tag{12.26}$$

A convenient differential form of the continuity equation can be obtained from Eq. 12.25a, in the form

$$\rho AV = \text{constant}$$

Differentiating and dividing by ρAV yields

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{12.27}$$

Solving Eq. 12.27 for dA/A gives

$$\frac{dA}{A} = -\frac{dV}{V} - \frac{d\rho}{\rho}$$

Substituting from Eq. 12.26 gives

$$\frac{dA}{A} = \frac{dp}{\rho V^2} - \frac{d\rho}{\rho}$$

or

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{dp/d\rho}\right]$$

Now recall that for an isentropic process, $dp/d\rho = \partial p/\partial\rho)_s = c^2$, so

$$\frac{dA}{A} = \frac{dp}{\rho V^2}\left[1 - \frac{V^2}{c^2}\right] = \frac{dp}{\rho V^2}[1 - M^2]$$

or

$$\frac{dp}{\rho V^2} = \frac{dA}{A}\frac{1}{[1 - M^2]} \tag{12.28}$$

Substituting from Eq. 12.26 into Eq. 12.28, we obtain

$$\frac{dV}{V} = -\frac{dA}{A}\frac{1}{[1 - M^2]} \tag{12.29}$$

Note that for an isentropic flow there can be no friction. Equations 12.28 and 12.29 confirm that for this case, from a momentum point of view we expect an increase in pressure to cause a decrease in speed, and vice versa. Although we cannot use them for computations because we have not so far determined how M varies with A, Eqs. 12.28 and 12.29 give us very interesting insights into how the pressure and velocity change as we change the area of the flow. Three possibilities are discussed below.

Subsonic Flow, $M < 1$

For $M < 1$, the factor $1/[1 - M^2]$ in Eqs. 12.28 and 12.29 is positive, so that a positive dA leads to a positive dp and a negative dV. These mathematical results mean that in a *divergent* section $(dA > 0)$ the flow must experience an *increase* in pressure $(dp > 0)$ and the velocity must *decrease*$(dV < 0)$. Hence *a divergent channel is a subsonic diffuser* that decelerates a flow.

On the other hand, a negative dA leads to a negative dp and a positive dV. These mathematical results mean that in a *convergent* section $(dA < 0)$ the flow must experience a *decrease* in pressure $(dp < 0)$ and the velocity must *increase*$(dV > 0)$. Hence *a convergent channel is a subsonic nozzle* that accelerates a flow.

These results are consistent with our everyday experience and are not surprising—for example, recall the venturi meter in Chapter 8, in which a reduction in area at the throat of the venturi led to a local increase in velocity, and because of the Bernoulli principle, to a pressure drop, and the divergent

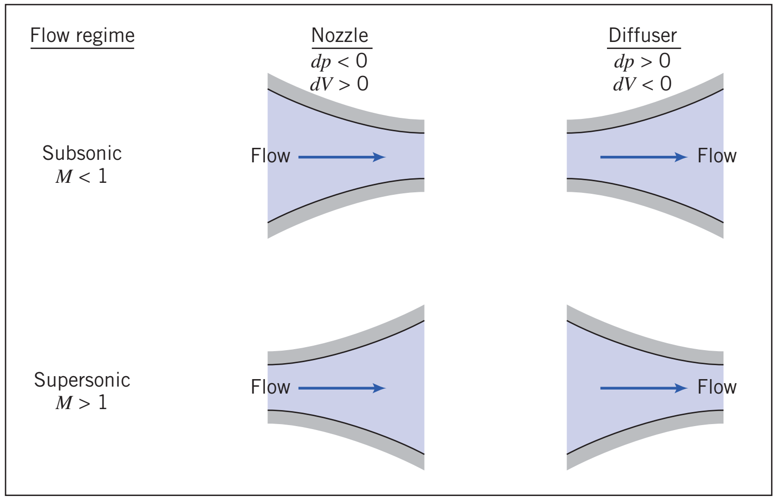


Fig. 12.8 Nozzle and diffuser shapes as a function of initial Mach number.

section led to pressure recovery and flow deceleration. (The Bernoulli principle assumes incompressible flow, which is the limiting case of subsonic flow.) The subsonic diffuser and nozzle are also shown in Fig. 12.8.

Supersonic Flow, $M > 1$

For $M > 1$, the factor $1/[1-M^2]$ in Eqs. 12.28 and 12.29 is negative, so that a positive dA leads to a negative dp and a positive dV. These mathematical results mean that in a *divergent* section $(dA > 0)$ the flow must experience a *decrease* in pressure $(dp < 0)$ and the velocity must *increase*$(dV > 0)$. Hence *a divergent channel is a supersonic nozzle*.

On the other hand, a negative dA leads to a positive dp and a negative dV. These mathematical results mean that in a *convergent* section $(dA < 0)$ the flow must experience an *increase* in pressure $(dp > 0)$ and the velocity must *decrease*$(dV < 0)$. Hence *a convergent channel is a supersonic diffuser*.

These results are inconsistent with our everyday experience and are at first a bit surprising—they are the opposite of what we saw in the venturi meter! The results are consistent with the laws of physics; for example, an increase in pressure must lead to a flow deceleration because pressure forces are the only forces acting. The supersonic nozzle and diffuser are also shown in Fig. 12.8.

These somewhat counterintuitive results can be understood when we realize that we are used to assuming that $\rho = \text{constant}$, but we are now in a flow regime where the fluid density is a function of flow conditions. From Eq. 12.27,

$$\frac{dV}{V} = -\frac{dA}{A} - \frac{d\rho}{\rho}$$

For example, in a supersonic diverging flow (dA positive), the flow actually accelerates (dV also positive) because the density drops sharply ($d\rho$ is negative and large, with the net result that the right side of the equation is positive). We can see examples of supersonic diverging nozzles in the space shuttle main engines, each of which has a nozzle about 3 m long with an 2.4 m exit diameter. The maximum thrust is obtained from the engines when the combustion gases exit at the highest possible speed, which the nozzles achieve.

Sonic Flow, $M = 1$

As we approach $M = 1$, from either a subsonic or supersonic state, the factor $1/[1-M^2]$ in Eqs. 12.28 and 12.29 approaches infinity, implying that the pressure and velocity changes also approach infinity. This is obviously unrealistic, so we must look for some other way for the equations to make physical sense. The only way we can avoid these singularities in pressure and velocity is if we require that $dA \to 0$ as $M \to 1$.

Hence, for an isentropic flow, sonic conditions can only occur where the area is constant! We can be even more specific: We can imagine approaching $M = 1$ from either a subsonic or a supersonic state. A subsonic flow $(M < 1)$ would need to be accelerated using a subsonic nozzle, which we have learned is a converging section; a supersonic flow $(M > 1)$ would need to be decelerated using a supersonic diffuser, which is also a converging section. Hence, sonic conditions are limited not just to a location of constant area but one that is a minimum area. The important result is that *for isentropic flow the sonic condition $M = 1$ can only be attained at a throat or section of minimum area*. This does *not* mean that a throat *must* have $M = 1$. After all, we may have a low speed flow or even no flow at all in the device!

We can see that to isentropically accelerate a fluid from rest to supersonic speed we would need to have a subsonic nozzle (converging section) followed by a supersonic nozzle (diverging section), with $M = 1$ at the throat. This device is called a *converging-diverging nozzle* (C-D nozzle). Of course, to create a supersonic flow we need more than just a C-D nozzle: We must also generate and maintain a pressure difference between the inlet and exit. We will discuss shortly C-D nozzles in some detail, and the pressures required to accomplish a change from subsonic to supersonic flow.

We must be careful in our discussion of isentropic flow, especially deceleration, because real fluids can experience nonisentropic phenomena such as boundary-layer separation and shock waves. In practice, supersonic flow cannot be decelerated to exactly $M = 1$ at a throat because sonic flow near a throat is unstable in a rising (adverse) pressure gradient. It turns out that disturbances that are always present in a real subsonic flow propagate upstream, disturbing the sonic flow at the throat, causing shock waves to form and travel upstream, where they may be disgorged from the inlet of the supersonic diffuser.

The throat area of a real supersonic diffuser must be slightly larger than that required to reduce the flow to $M = 1$. Under the proper downstream conditions, a weak normal shock forms in the diverging channel just downstream from the throat. Flow leaving the shock is subsonic and decelerates in the diverging channel. Thus, deceleration from supersonic to subsonic flow cannot occur isentropically in practice, since the weak normal shock causes an entropy increase. Normal shocks will be analyzed in Section 12.7.

For accelerating flows (favorable pressure gradients), the idealization of isentropic flow is generally a realistic model of the actual flow behavior. For decelerating flows, the idealization of isentropic flow may not be realistic because of the adverse pressure gradients and the attendant possibility of flow separation, as discussed for incompressible boundary-layer flow in Chapter 9.

Reference Stagnation and Critical Conditions for Isentropic Flow of an Ideal Gas

As we mentioned at the beginning of this section, in principle we could use Eq. 12.25 to analyze one-dimensional isentropic flow of an ideal gas, but the computations would be somewhat tedious. Instead, because the flow is isentropic, we can use the results of Sections 12.3 (reference stagnation conditions) and 12.4 (reference critical conditions). The idea is illustrated in Fig. 12.9: Instead of using Eq. 12.25 to compute, for example, properties at state ② from those at state ①, we can use state ① to determine two reference states (the stagnation state and the critical state) and then use these to obtain properties at state ②. We need two reference states because the reference stagnation state does not provide area information (mathematically the stagnation area is infinite).

We will use Eqs. 12.21 (renumbered for convenience),

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{12.30a}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \tag{12.30b}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)} \tag{12.30c}$$

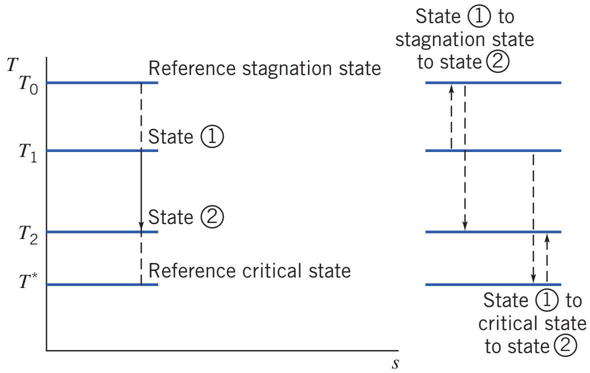


Fig. 12.9 Example of stagnation and critical reference states in the *Ts* plane.

We note that *the stagnation conditions are constant throughout the isentropic flow*. The critical conditions ($M = 1$) were related to stagnation conditions in Section 12.4,

$$\frac{p_0}{p*} = \left[\frac{k+1}{2}\right]^{k/(k-1)} \tag{12.22a}$$

$$\frac{T_0}{T*} = \frac{k+1}{2} \tag{12.22b}$$

$$\frac{\rho_0}{\rho^*} = \left[\frac{k+1}{2}\right]^{1/(k-1)} \tag{12.22c}$$

$$V^* = c^* = \sqrt{\frac{2k}{k+1}RT_0} \tag{12.23}$$

Although a particular flow may never attain sonic conditions, as in the example in Fig. 12.9, we will still find the critical conditions useful as reference conditions. Equations 12.30a, 12.30b, and 12.30c relate local properties (p, ρ, T, and V) to stagnation properties (p_0, ρ_0, and T_0) via the Mach number M, and Eqs. 12.22 and 12.23 relate critical properties (p^*, ρ^*, T^*, and V^*) to stagnation properties (p_0, ρ_0, and T_0), respectively, but we have yet to obtain a relation between areas A and A^*. To do this we start with continuity (Eq. 12.25a) in the form

$$\rho AV = \text{constant} = \rho^* A^* V^*$$

Then

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V} = \frac{\rho^*}{\rho}\frac{c^*}{Mc} = \frac{1}{M}\frac{\rho^*}{\rho}\sqrt{\frac{T^*}{T}}$$

$$\frac{A}{A^*} = \frac{1}{M}\frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\sqrt{\frac{T^*/T_0}{T/T_0}}$$

$$\frac{A}{A^*} = \frac{1}{M}\frac{\left[1+\frac{k-1}{2}M^2\right]^{1/(k-1)}}{\left[\frac{k+1}{2}\right]^{1/(k-1)}}\left[\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{1/2}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{k-1}{2}M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \tag{12.30d}$$

Equation 12.30 forms a set that is convenient for analyzing isentropic flow of an ideal gas with constant specific heats, which we usually use instead of the basic equations, Eq. 12.25. For convenience, we list Eq. 12.30 together:

$$\frac{p_0}{p} = \left[1+\frac{k-1}{2}M^2\right]^{k/(k-1)} \tag{12.30a}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2 \tag{12.30b}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{1/(k-1)} \tag{12.30c}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} \tag{12.30d}$$

Equation 12.30 provides property relations in terms of the local Mach number, the stagnation conditions, and critical conditions. These equations are readily programmed and there are also interactive websites that make them available (see, for example, [4]). These equations are also available in the *Excel* spreadsheets on the website, with add-in functions available for computing pressure, temperature, density, and area ratios from M, or computing M from the ratios. While they are somewhat complicated algebraically, they have the advantage over the basic equations, Eq. 12.25, that they are not coupled. Each property can be found directly from its stagnation value and the Mach number.

Equation 12.30d shows the relation between Mach number M and area A. The critical area A^* is used to normalize area A. For each Mach number M, we obtain a unique area ratio, but as shown in Fig 12.10 each A/A^* ratio (except 1) has two possible Mach numbers—one subsonic, the other supersonic. The shape shown in Fig. 12.10 *looks* like a converging–diverging section for accelerating from a subsonic to a supersonic flow (with, as necessary, $M = 1$ only at the throat), but in practice this is not the shape to which such a passage would be built. For example, the diverging section usually will have a much less severe angle of divergence to reduce the chance of flow separation.

Appendix D.1 lists flow functions for property ratios T_0/T, p_0/p, ρ_0/ρ, and A/A^* in terms of M for isentropic flow of an ideal gas. A table of values, as well as a plot of these property ratios, is presented for air ($k = 1.4$) for a limited range of Mach numbers. The associated *Excel* workbook, *Isentropic Relations*, can be used to print a larger table of values for air and other ideal gases.

Example 12.7 demonstrates use of some of the above equations. As shown in Fig. 12.9, we can use the equations to relate a property at one state to the stagnation value and then from the stagnation value to a second state, but note that we can accomplish this in one step—for example, p_2 can be obtained from p_1 by writing $p_2 = (p_2/p_0)(p_0/p_1)p_1$, where the pressure ratios come from Eq. 12.30a evaluated at the two Mach numbers.

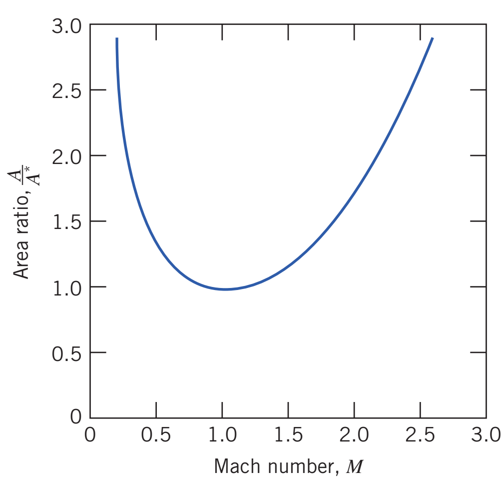


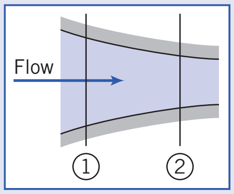
Fig. 12.10 Variation of A/A^* with Mach number for isentropic flow of an ideal gas with $k = 1.4$.

Example 12.7 ISENTROPIC FLOW IN A CONVERGING CHANNEL

Air flows isentropically in a channel. At section 1, the Mach number is 0.3, the area is 0.001 m^2, and the absolute pressure and the temperature are 650 kPa and 62°C, respectively. At section 2, the Mach number is 0.8. Sketch the channel shape, plot a Ts diagram for the process, and evaluate properties at section 2. Verify that the results agree with the basic equations, Eq. 12.25.

Given: Isentropic flow of air in a channel. At sections 1 and 2, the following data are given: $M_1 = 0.3$, $T_1 = 62°C$, $p_1 = 650$ kPa (abs), $A_1 = 0.001$ m^2, and $M_2 = 0.8$.

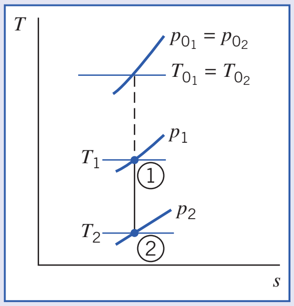
Find:
(a) The channel shape.
(b) A Ts diagram for the process.
(c) Properties at section 2.
(d) Show that the results satisfy the basic equations.



Solution: To accelerate a subsonic flow requires a converging nozzle. The channel shape must be as shown.

On the Ts plane, the process follows an $s = $ constant line. Stagnation conditions remain fixed for isentropic flow.

Consequently, the stagnation temperature at section 2 can be calculated (for air, $k = 1.4$) from Eq. 12.30b.



$$T_{0_2} = T_{0_1} = T_1\left[1 + \frac{k-1}{2}M_1^2\right]$$

$$= (62 + 273)\ \text{K}\left[1 + 0.2(0.3)^2\right]$$

$$T_{0_2} = T_{0_1} = 341\ \text{K} \longleftarrow T_{0_1}, T_{0_2}$$

For p_{0_2}, from Eq. 12.30a,

$$p_{0_2} = p_{0_1} = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)} = 650\ \text{kPa}[1 + 0.2(0.3)^2]^{3.5}$$

$$p_{0_2} = 692\ \text{kPa (abs)} \longleftarrow p_{0_2}$$

For T_2, from Eq. 12.30b,

$$T_2 = T_{0_2}\Big/\left[1 + \frac{k-1}{2}M_2^2\right] = 341\ \text{K}/[1 + 0.2(0.8)^2]$$

$$T_2 = 302\ \text{K} \longleftarrow T_2$$

For p_2, from Eq. 12.30a,

$$p_2 = p_{0_2}\Big/\left[1 + \frac{k-1}{2}M_2^2\right]^{k/k-1} = 692\ \text{kPa}/[1 + 0.2(0.8)^2]^{3.5}$$

$$p_2 = 454\ \text{kPa} \longleftarrow p_2$$

Note that we could have directly computed T_2 from T_1 because $T_0 = $ constant:

$$\frac{T_2}{T_1} = \frac{T_2}{T_0}\Big/\frac{T_0}{T_1} = \left[1 + \frac{k-1}{2}M_1^2\right]\Big/\left[1 + \frac{k-1}{2}M_2^2\right] = \left[1 + 0.2(0.3)^2\right]\Big/\left[1 + 0.2(0.8)^2\right]$$

$$\frac{T_2}{T_1} = \frac{0.8865}{0.9823} = 0.9025$$

Hence,

$$T_2 = 0.9025\ T_1 = 0.9025(273 + 62)\text{K} = 302\ \text{K}$$

Similarly, for p_2,

$$\frac{p_2}{p_1} = \frac{p_2}{p_0}\Big/\frac{p_0}{p_1} = 0.8865^{3.5}\Big/0.9823^{3.5} = 0.6982$$

Hence,

$$p_2 = 0.6982\, p_1 = 0.6982(650 \text{ kPa}) = 454 \text{ kPa}$$

The density ρ_2 at section ② can be found from Eq. 12.30c using the same procedure we used for T_2 and p_2, or we can use the ideal gas equation of state, Eq. 12.25e,

$$\rho_2 = \frac{p_2}{RT_2} = 4.54 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287 \text{ N}\cdot\text{m}} \times \frac{1}{302 \text{ K}} = 5.24 \text{ kg/m}^3 \longleftarrow \rho_2$$

and the velocity at section ② is

$$V_2 = M_2 c_2 = M_2\sqrt{kRT_2} = 0.8 \times \sqrt{1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 302 \text{ K} \times \frac{\text{kg}\cdot\text{m}}{\text{s}^2\cdot\text{N}}} = 279 \text{ m/s} \longleftarrow V_2$$

The area A_2 can be computed from Eq. 12.30d, noting that A^* is constant for this flow,

$$\frac{A_2}{A_1} = \frac{A_2 A^*}{A^* A_1} = \frac{1}{M_2}\left[\frac{1+\frac{k-1}{2}M_2^2}{\frac{k+1}{2}}\right]^{(k+2)/2(k-1)} \Bigg/ \frac{1}{M_1}\left[\frac{1+\frac{k-1}{2}M_1^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

$$= \frac{1}{0.8}\left[\frac{1+0.2(0.8)^2}{1.2}\right]^3 \Bigg/ \frac{1}{0.3}\left[\frac{1+0.2(0.3)^2}{1.2}\right]^3 = \frac{1.038}{2.035} = 0.5101$$

Hence,

$$A_2 = 0.5101 A_1 = 0.5101(0.001 \text{ m}^2) = 5.10 \times 10^{-4} \text{ m}^2 \longleftarrow A_2$$

Note that $A_2 < A_1$ as expected.

Let us verify that these results satisfy the basic equations.

We first need to obtain ρ_1 and V_1:

$$\rho_1 = \frac{p_1}{RT_1} = 6.5 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287 \text{ N}\cdot\text{m}} \times \frac{1}{335 \text{ K}} = 6.76 \text{ kg/m}^3$$

and

$$V_1 = M_1 c_1 = M_1\sqrt{kRT_1} = 0.3 \times \sqrt{1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 335 \text{ K} \times \frac{\text{kg}\cdot\text{m}}{\text{s}^2\cdot\text{N}}} = 110 \text{ m/s}$$

The mass conservation equation is

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \tag{12.25a}$$

$$\dot{m} = 6.76 \frac{\text{kg}}{\text{m}^3} \times 110 \frac{\text{m}}{\text{s}} \times 0.001 \text{ m}^2 = 5.24 \frac{\text{kg}}{\text{m}^3} \times 279 \frac{\text{m}}{\text{s}} \times 0.00051 \text{ m}^2 = 0.744 \text{ kg/s} \qquad \text{(Check!)}$$

We cannot check the momentum equation (Eq. 12.25b) because we do not know the force R_x produced by the walls of the device (we could use Eq. 12.25b to compute this if we wished). The energy equation is

$$h_{0_1} = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{0_2} = h_0 \tag{12.25c}$$

We will check this by replacing enthalpy with temperature using Eq. 13.2f,

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p(T_2 - T_1) \tag{12.25f}$$

so the energy equation becomes

$$c_p T_1 + \frac{V_1^2}{2} = c_p T_2 + \frac{V_2^2}{2} = c_p T_0$$

Using c_p for air from Table A.6,

$$c_pT_1 + \frac{V_1^2}{2} = 1004\frac{\text{J}}{\text{kg}\cdot\text{K}} \times 335\ \text{K} + \frac{(110)^2}{2}\left(\frac{\text{m}}{\text{s}}\right)^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 342\ \text{kJ/kg}$$

$$c_pT_2 + \frac{V_2^2}{2} = 1004\frac{\text{J}}{\text{kg}\cdot\text{K}} \times 302\ \text{K} + \frac{(278)^2}{2}\left(\frac{\text{m}}{\text{s}}\right)^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \times \frac{\text{J}}{\text{N}\cdot\text{m}} = 342\ \text{kJ/kg}$$

$$c_pT_0 = 1004\frac{\text{J}}{\text{kg}\cdot\text{K}} \times 341\ \text{K} = 342\ \text{kJ/kg} \qquad \text{(Check!)}$$

The final equation we can check is the relation between pressure and density for an isentropic process (Eq. 12.25g),

$$\frac{p_1}{\rho_1^k} = \frac{p_2}{\rho_2^k} = \frac{p}{\rho^k} = \text{constant} \qquad \text{(Check!)}$$

$$\frac{p_1}{\rho_1^{1.4}} = \frac{650\ \text{kPa}}{\left(6.76\ \dfrac{\text{kg}}{\text{m}^3}\right)^{1.4}} = \frac{p_2}{\rho_2^{1.4}} = \frac{454\ \text{kPa}}{\left(5.24\ \dfrac{\text{kg}}{\text{m}^3}\right)^{1.4}} = 44.7\frac{\text{kPa}}{\left(\dfrac{\text{kg}}{\text{m}^3}\right)^{1.4}} \qquad \text{(Check!)}$$

The basic equations are satisfied by our solution.

This problem illustrates:

- Use of the isentropic equations, Eq. 12.30
- That the isentropic equations are consistent with the basic equations, Eq. 12.25
- That the computations can be quite laborious without using preprogrammed isentropic relations (available, for example, in the *Excel* add-ins on the website)!

The *Excel* workbook for this example is convenient for performing the calculations, using either the isentropic equations or the basic equations.

Isentropic Flow in a Converging Nozzle

Now that we have our computing equations (Eq. 12.30) for analyzing isentropic flows, we are ready to see how we could obtain flow in a nozzle, starting from rest. We first look at the converging nozzle, and then the C-D nozzle. In either case, to produce a flow we must provide a pressure difference. For example, as illustrated in the converging nozzle shown in Fig. 12.11*a*, we can do this by providing the gas from a reservoir (or "plenum chamber") at p_0 and T_0, and using a vacuum pump/valve combination to create a low pressure, the "back pressure," p_b. We are interested in what happens to the gas properties as the gas flows through the nozzle, and also in knowing how the mass flow rate increases as we progressively lower the back pressure.

Let us call the pressure at the exit plane p_e. We will see that this will often be equal to the applied back pressure, p_b, but not always! The results we obtain as we progressively open the valve from a closed position are shown in Figs. 12.11*b* and 12.11*c*. We consider each of the cases shown.

When the valve is closed, there is no flow through the nozzle. The pressure is p_0 throughout, as shown by condition (i) in Fig. 12.11*a*.

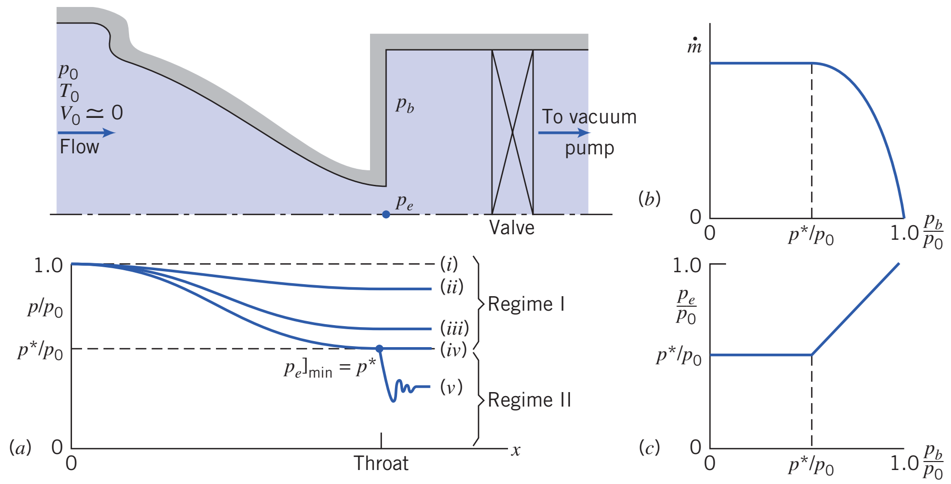


Fig. 12.11 Converging nozzle operating at various back pressures.

If the back pressure, p_b, is now reduced to slightly less than p_0, there will be flow through the nozzle with a decrease in pressure in the direction of flow, as shown by condition (ii). Flow at the exit plane will be subsonic with the exit-plane pressure equal to the back pressure.

What happens as we continue to decrease the back pressure? As expected, the flow rate will continue to increase, and the exit-plane pressure will continue to decrease, as shown by condition (iii) in Fig. 12.11*a*.

As we progressively lower the back pressure the flow rate increases, and hence, so do the velocity and Mach number at the exit plane. The question arises: "Is there a limit to the mass flow rate through the nozzle?" or, to put it another way, "Is there an upper limit on the exit Mach number?" The answer to these questions is "Yes!" To see this, recall that for isentropic flow Eq. 12.29 applies:

$$\frac{dV}{V} = -\frac{dA}{A}\frac{1}{[1-M^2]} \tag{12.29}$$

From this we learned that the *only* place we can have sonic conditions $(M=1)$ is where the change in area dA is zero. We *cannot* have sonic conditions anywhere in the converging section. Logically we can see that the *maximum exit Mach number is one*. Because the flow started from rest $(M=0)$, if we had $M>1$ at the exit, we would have had to pass through $M=1$ somewhere in the converging section, which would be a violation of Eq. 12.29.

Hence, the maximum flow rate occurs when we have sonic conditions at the exit plane, when $M_e=1$, and $p_e=p_b=p^*$, the critical pressure. This is shown as condition (iv) in Fig. 12.11*a* and is called a "choked flow," beyond which the flow rate cannot be increased. From Eq. 12.30a with $M=1$ (or from Eq. 12.21a),

$$\left.\frac{p_e}{p_0}\right|_{\text{choked}} = \frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \tag{12.31}$$

For air, $k=1.4$, so $p_e/p_0]_{\text{choked}}=0.528$. For example, if we wish to have sonic flow at the exit of a nozzle from a plenum chamber that is at atmospheric pressure, we would need to maintain a back pressure of about 7.76 psia or about 6.94 psig vacuum. This does not sound difficult for a vacuum pump to generate but actually takes a lot of power to maintain because we will have a large mass flow rate through the pump. For the maximum, or choked, mass flow rate we have

$$\dot{m}_{\text{choked}} = \rho^* V^* A^*$$

Using the ideal gas equation of state, Eq. 12.25e, and the stagnation to critical pressure and temperature ratios, Eq. 12.30a and 12.30b, respectively, with $M=1$ (or Eq. 12.21a and 12.21b, respectively), with $A^*=A_e$, it can be shown that this becomes

$$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \tag{12.32a}$$

Note that for a given gas (k and R), the maximum flow rate in the converging nozzle depends *only* on the size of the exit area (A_e) and the conditions in the reservoir (p_0, T_0).

For air, for convenience, we write an "engineering" form of Eq. 12.32a,

$$\dot{m}_{\text{choked}} = 0.04\,\frac{A_e p_0}{\sqrt{T_0}} \tag{12.32b}$$

with $\dot{m}_{\text{choked}}$ in kg/s, A_e in m^2, p_0 in Pa, and T_0 in K.

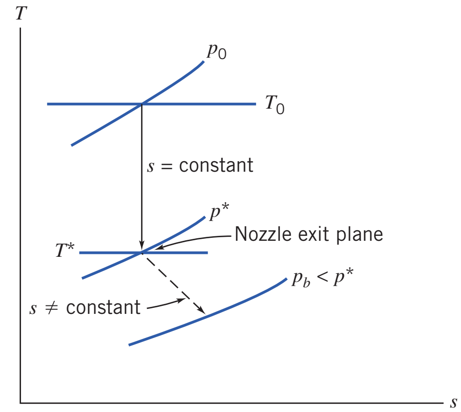


Fig. 12.12 Schematic *Ts* diagram for choked flow through a converging nozzle.

Suppose we now insist on lowering the back pressure below this "benchmark" level of p^*. Our next question is "What will happen to the flow in the nozzle?" The answer is "Nothing!" The flow remains choked: The mass flow rate does not increase, as shown in Fig. 12.11b, and the pressure distribution in the nozzle remains unchanged, with $p_e = p^* > p_b$, as shown in condition (v) in Fig. 12.11a and 12.11c. After exiting, the flow adjusts down to the applied back pressure, but does so in a nonisentropic, three-dimensional manner in a series of expansion waves and shocks, and for this part of the flow our one-dimensional, isentropic flow concepts no longer apply. We will return to this discussion in Section 12.8.

This idea of choked flow seems a bit strange, but can be explained in at least two ways. First, we have already discussed that to increase the mass flow rate beyond choked would require $M_e > 1$, which is not possible. Second, once the flow reaches sonic conditions, it becomes "deaf" to downstream conditions: Any change (i.e., a reduction) in the applied back pressure propagates in the fluid at the speed of sound in all directions, so it gets "washed" downstream by the fluid which is moving at the speed of sound at the nozzle exit.

Flow through a converging nozzle may be divided into two regimes:

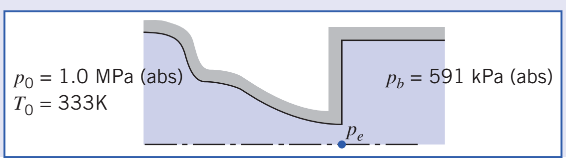
1 In Regime I, $1 \geq p_b/p_0 \geq p^*/p_0$. Flow to the throat is isentropic and $p_e = p_b$.
2 In Regime II, $p_b/p_0 < p^*/p_0$. Flow to the throat is isentropic, and $M_e = 1$. A nonisentropic expansion occurs in the flow leaving the nozzle and $p_e = p^* > p_b$ (entropy increases because this is adiabatic but irreversible).

Although isentropic flow is an idealization, it often is a very good approximation for the actual behavior of nozzles. Since a nozzle is a device that accelerates a flow, the internal pressure gradient is favorable. This tends to keep the wall boundary layers thin and to minimize the effects of friction. The flow processes corresponding to Regime II are shown on a *Ts* diagram in Fig. 12.12. Two problems involving converging nozzles are solved in Examples 12.8 and 12.9.

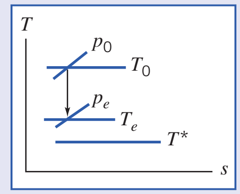
Example 12.8 ISENTROPIC FLOW IN A CONVERGING NOZZLE

A converging nozzle, with a throat area of 0.001 m^2, is operated with air at a back pressure of 591 kPa (abs). The nozzle is fed from a large plenum chamber where the absolute stagnation pressure and temperature are 1.0 MPa and 60°C. The exit Mach number and mass flow rate are to be determined.

Given: Air flow through a converging nozzle at the conditions shown: Flow is isentropic.



Find: (a) M_e.
(b) $\dot{m}$.



Solution: The first step is to check for choking. The pressure ratio is

$$\frac{p_b}{p_0} = \frac{5.91 \times 10^5}{1.0 \times 10^6} = 0.591 > 0.528$$

so the flow is *not* choked. Thus $p_b = p_e$, and the flow is isentropic, as sketched on the Ts diagram.

Since $p_0 = \text{constant}$, M_e may be found from the pressure ratio,

$$\frac{p_0}{p_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/(k-1)}$$

Solving for M_e, since $p_e = p_b$, we obtain

$$1 + \frac{k-1}{2} M_e^2 = \left(\frac{p_0}{p_b}\right)^{(k-1)/k}$$

and

$$M_e = \left\{\left[\left(\frac{p_0}{p_b}\right)^{(k-1)/k} - 1\right]\frac{2}{k-1}\right\}^{1/2} = \left\{\left[\left(\frac{1.0 \times 10^6}{5.91 \times 10^5}\right)^{0.286} - 1\right]\frac{2}{1.4-1}\right\}^{1/2} = 0.90 \longleftarrow M_e$$

The mass flow rate is

$$\dot{m} = \rho_e V_e A_e = \rho_e M_e c_e A_e$$

We need T to find ρ_e and c_e. Since $T_0 = \text{constant}$,

$$\frac{T_0}{T_e} = 1 + \frac{k-1}{2} M_e^2$$

or

$$T_e = \frac{T_0}{1 + \dfrac{k-1}{2} M_e^2} = \frac{(273+60)\,\text{K}}{1 + 0.2(0.9)^2} = 287\ \text{K}$$

$$c_e = \sqrt{kRT_e} = \left[1.4 \times 287 \frac{\text{N} \cdot \text{m}}{\text{kg} \cdot \text{K}} \times 287\ \text{K} \times \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2}\right]^{1/2} = 340\ \text{m/s}$$

and

$$\rho_e = \frac{p_e}{RT_e} = 5.91 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg} \cdot \text{K}}{287\ \text{N} \cdot \text{m}} \times \frac{1}{287\ \text{K}} = 7.18\ \text{kg/m}^3$$

Finally,

$$\dot{m} = \rho_e M_e c_e A_e = 7.18 \frac{\text{kg}}{\text{m}^3} \times 0.9 \times 340 \frac{\text{m}}{\text{s}} \times 0.0001\ \text{m}^2$$

$$= 2.20\ \text{kg/s} \longleftarrow \dot{m}$$

This problem illustrates use of the isentropic equations, Eqs. 12.30a for a flow that is not choked.

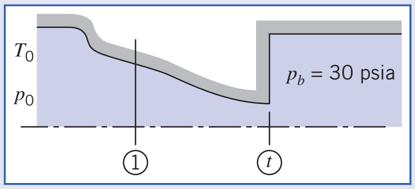
The *Excel* workbook for this example is convenient for performing the calculations (using either the isentropic equations or the basic equations). (The *Excel* add-ins for isentropic flow, on the website, also make calculations much easier.)

Example 12.9 CHOKED FLOW IN A CONVERGING NOZZLE

Air flows isentropically through a converging nozzle. At a section where the nozzle area is 0.0012 m^2, the local pressure, temperature, and Mach number are 413.4 kPa, 4.5°C, and 0.52, respectively. The back pressure is 206.7 kPa (abs). The Mach number at the throat, the mass flow rate, and the throat area are to be determined.

Given: Air flow through a converging nozzle at the conditions shown:

$$M_1 = 0.52$$
$$T_1 = 4.5°\text{C}$$
$$p_1 = 413.4 \text{ kPa (abs)}$$
$$A_1 = 0.0012 \text{ m}^2$$



Find: (a) M_t. (b) $\dot{m}$. (c) A_t.

Solution:

First we check for choking, to determine if flow is isentropic down to p_b. To check, we evaluate the stagnation conditions.

$$p_0 = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k/(k-1)} = 413.4 \text{ kPa (abs)} \left[1 + 0.2(0.52)^2\right]^{3.5} = 497.1 \text{ kPa (abs)}$$

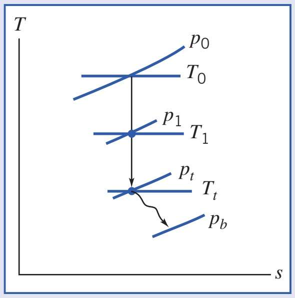
The back pressure ratio is

$$\frac{p_b}{p_0} = \frac{206.7}{497.1} = 0.416 < 0.528$$

so the flow is choked! For choked flow,

$$M_t = 1.0 \longleftarrow M_t$$

The Ts diagram is



The mass flow rate may be found from conditions at section ①, using $\rho_1 V_1 A_1$.

$$V_1 = M_1 c_1 = M_1\sqrt{kRT_1}$$

$$= 0.52\left[1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times (273 + 4.5) \text{ K} \times 32.3 \frac{\text{lbm}}{\text{slug}} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2}$$

$$V_1 = 570 \text{ ft/s}$$

$$\rho_1 = \frac{p_1}{RT_1} = 413.4 \times 1000 \text{ Pa} \times \frac{\text{kg}\cdot\text{K}}{287 \text{ N}\cdot\text{m}} \times (1/277.5 \text{ K}) \times \frac{\text{N}}{\text{Pa}\cdot\text{m}^2} = 5.19 \text{ (kg/m}^3)$$

$$\dot{m} = \rho_1 V_1 A_1 = 5.19 \text{ (kg/m}^3) \times 173.6 \text{ (m/s)} \times 0.0012 \text{ m}^2 = 1.08 \text{ (kg/s)} \longleftarrow \dot{m}$$

From Eq. 12.29,

$$\frac{A_1}{A^*} = \frac{1}{M_1}\left[\frac{1 + \frac{k-1}{2}M_1^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} = \frac{1}{0.52}\left[\frac{1 + 0.2(0.52)^2}{1.2}\right]^{3.00} = 1.303$$

For choked flow, $A_t = A^*$. Thus,

$$A_t = A^* = \frac{A_1}{1.303} = \frac{0.0012 \text{ m}^2}{1.303}$$

$$A_t = 9.21 \times 10^{-4} \text{ m}^2 \longleftarrow A_t$$

This problem illustrates use of the isentropic equations, Eq. 12.30a for a flow that is choked.

- Because the flow is choked, we could also have used Eq. 12.32a for $\dot{m}$ (after finding T_0).

The *Excel* workbook for this example is convenient for performing the calculations. (The *Excel* add-ins for isentropic flow, on the website, also make calculations much easier.)

Isentropic Flow in a Converging-Diverging Nozzle

Having considered isentropic flow in a converging nozzle, we turn now to isentropic flow in a converging-diverging (C-D) nozzle. As in the previous case, flow through the converging-diverging passage of Fig. 12.13 is induced by a vacuum pump downstream and is controlled by the valve shown; upstream stagnation conditions are constant. Pressure in the exit plane of the nozzle is p_e; the nozzle discharges to back pressure p_b. As for the converging nozzle, we wish to see, among other things, how the flow rate varies with the driving force, the applied pressure difference $(p_0 - p_b)$. Consider the effect of gradually reducing the back pressure. The results are illustrated graphically in Fig. 12.13. Let us consider each of the cases shown.

With the valve initially closed, there is no flow through the nozzle; the pressure is constant at p_0. Opening the valve slightly (p_b slightly less than p_0) produces pressure distribution curve (i). If the flow rate is low enough, the flow will be subsonic and essentially incompressible at all points on this curve. Under these conditions, the C-D nozzle will behave as a venturi, with flow accelerating in the converging portion until a point of maximum velocity and minimum pressure is reached at the throat, then decelerating in the diverging portion to the nozzle exit. This behavior is described accurately by the Bernoulli equation, Eq. 6.8.

As the valve is opened farther and the flow rate is increased, a more sharply defined pressure minimum occurs, as shown by curve (ii). Although compressibility effects become important, the flow is still subsonic everywhere, and flow decelerates in the diverging section. Finally, as the valve is opened farther, curve (iii) results. At the section of minimum area, the flow finally reaches $M = 1$, and the nozzle is choked—the flow rate is the maximum possible for the given nozzle and stagnation conditions.

All flows with pressure distributions (i), (ii), and (iii) are isentropic; as we progress from (i) to (ii) to (iii) we are generating increasing mass flow rates. Finally, when curve (iii) is reached, critical conditions are present at the throat. For this flow rate, the flow is choked, and

$$\dot{m} = \rho^* V^* A^*$$

where $A^* = A_t$, just as it was for the converging nozzle, and for this maximum possible flow rate Eq. 12.32a applies (with A_e replaced with the throat area A_t),

$$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)} \tag{12.33a}$$

Note that for a given gas (k and R), the maximum flow rate in the C-D nozzle depends *only* on the size of the throat area (A_t) and the conditions in the reservoir (p_0, T_0).

As with the converging nozzle, for air we write an "engineering" form of Eq. 12.33a,

$$\dot{m}_{\text{choked}} = 0.04 \frac{A_t p_0}{\sqrt{T_0}} \tag{12.33b}$$

with $\dot{m}_{\text{choked}}$ in kg/s, A_t in m^2, p_0 in Pa, and T_0 in K.

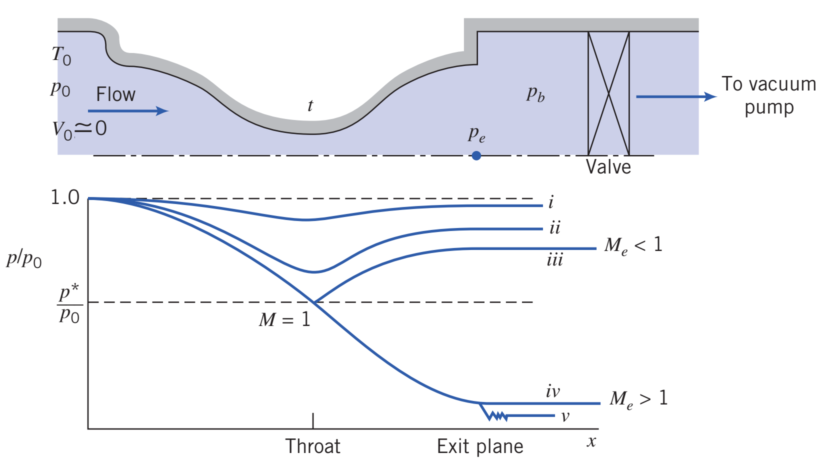


Fig. 12.13 Pressure distributions for isentropic flow in a converging-diverging nozzle.

with $\dot{m}_{\text{choked}}$ in lbm/s, A_t in ft^2, p_0 in psia, and T_0 in °R. We again have Eq. 12.32b and 12.32c, with the exit area A_e now replaced by the throat area A_t.

Any attempt to increase the flow rate by further lowering the back pressure will fail, for the two reasons we discussed earlier: once we attain sonic conditions, downstream changes can no longer be transmitted upstream; and we cannot exceed sonic conditions at the throat, because this would require passing through the sonic state somewhere in the converging section, which is not possible in isentropic flow.

With sonic conditions at the throat, we consider what *can* happen to the flow in the diverging section. We have previously discussed (see Fig. 12.8) that a diverging section will decelerate a subsonic flow $(M<1)$ but will accelerate a supersonic flow $(M>1)$—very different behaviors! The question arises: "Does a sonic flow behave as a subsonic or as a supersonic flow as it enters a diverging section?" The answer to this question is that it can behave like either one, depending on the downstream pressure! We have already seen subsonic flow behavior [curve (iii)]: the applied back pressure leads to a gradual downstream pressure increase, decelerating the flow. We now consider accelerating the choked flow.

To accelerate flow in the diverging section requires a pressure decrease. This condition is illustrated by curve (iv) in Fig. 12.13. The flow will accelerate isentropically in the nozzle provided the exit pressure is set at p_{iv}. Thus, we see that with a throat Mach number of unity, there are two possible isentropic flow conditions in the converging-diverging nozzle. This is consistent with the results of Fig. 12.10, where we found two Mach numbers for each A/A^* in isentropic flow.

Lowering the back pressure below condition (iv), say to condition (v), has no effect on flow in the nozzle. The flow is isentropic from the plenum chamber to the nozzle exit [as in condition (iv)] and then it undergoes a three-dimensional irreversible expansion to the lower back pressure. A nozzle operating under these conditions is said to be *underexpanded*, since additional expansion takes place outside the nozzle.

A converging-diverging nozzle generally is intended to produce supersonic flow at the exit plane. If the back pressure is set at p_{iv}, flow will be isentropic through the nozzle, and supersonic at the nozzle exit. Nozzles operating at $p_b = p_{iv}$ [corresponding to curve (iv) in Fig. 12.13] are said to operate at *design conditions*.

Flow leaving a C-D nozzle is supersonic when the back pressure is at or below nozzle design pressure. The exit Mach number is fixed once the area ratio, A_e/A^*, is specified. All other exit plane properties are uniquely related to stagnation properties by the fixed exit plane Mach number. The assumption of isentropic flow for a real nozzle at design conditions is a reasonable one. However, the one-dimensional flow model is inadequate for the design of relatively short nozzles.

Rocket-propelled vehicles use C-D nozzles to accelerate the exhaust gases to the maximum possible speed to produce high thrust. A propulsion nozzle is subject to varying ambient conditions during flight through the atmosphere, so it is impossible to attain the maximum theoretical thrust over the complete operating range. Because only a single supersonic Mach number can be obtained for each area ratio, nozzles for developing supersonic flow in wind tunnels often are built with interchangeable test sections, or with variable geometry.

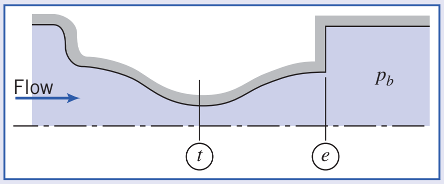
You probably have noticed that nothing has been said about the operation of converging-diverging nozzles with back pressure in the range $p_{iii} > p_b > p_{iv}$. For such cases, the flow cannot expand isentropically to p_b. Under these conditions, a shock (which may be treated as an irreversible discontinuity involving entropy increase) occurs somewhere within the flow. Following a discussion of normal shocks in Section 12.7, we shall complete the discussion of converging-diverging nozzle flows in Section 12.8.

Nozzles operating with $p_{iii} > p_b > p_{iv}$ are said to be *overexpanded* because the pressure at some point in the nozzle is less than the back pressure. Obviously, an overexpanded nozzle could be made to operate at a new design condition by removing a portion of the diverging section. In Example 12.10, we consider isentropic flow in a C-D nozzle and in Example 12.11, we consider choked flow in a C-D nozzle.

Example 12.10 ISENTROPIC FLOW IN A CONVERGING-DIVERGING NOZZLE

Air flows isentropically in a converging-diverging nozzle, with exit area of 0.001 m^2. The nozzle is fed from a large plenum where the stagnation conditions are 350 K and 1.0 MPa (abs). The exit pressure is 954 kPa (abs) and the Mach number at the throat is 0.68. Fluid properties and area at the nozzle throat and the exit Mach number are to be determined.

Given: Isentropic flow of air in C-D nozzle as shown:



$$T_0 = 350 \text{ K}$$
$$p_0 = 1.0 \text{ MPa (abs)}$$
$$p_b = 954 \text{ kPa (abs)}$$
$$M_t = 0.68 \quad A_e = 0.001 \text{ m}^2$$

Find: (a) Properties and area at nozzle throat.
(b) M_e.

Solution: Stagnation temperature is constant for isentropic flow. Thus, since

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$$

then

$$T_t = \frac{T_0}{1 + \frac{k-1}{2} M_t^2} = \frac{350 \text{ K}}{1 + 0.2(0.68)^2} = 320 \text{ K} \longleftarrow T_t$$

Also, since p_0 is constant for isentropic flow, then

$$p_t = p_0 \left(\frac{T_t}{T_0}\right)^{k/(k-1)} = p_0 \left[\frac{1}{1 + \frac{k-1}{2} M_t^2}\right]^{k/(k-1)}$$

$$p_t = 1.0 \times 10^6 \text{ Pa} \left[\frac{1}{1 + 0.2(0.68)^2}\right]^{3.5} = 734 \text{ kPa (abs)} \longleftarrow p_t$$

so

$$\rho_t = \frac{p_t}{RT_t} = 7.34 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg} \cdot \text{K}}{287 \text{ N} \cdot \text{m}} \times \frac{1}{320 \text{ K}} = 7.99 \text{ kg/m}^3 \longleftarrow \rho_t$$

and

$$V_t = M_t c_t = M_t \sqrt{kRT_t}$$
$$V_t = 0.68 \left[14 \times 287 \frac{\text{N} \cdot \text{m}}{\text{kg} \cdot \text{K}} \times 320 \text{ K} \times \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2}\right]^{1/2} = 244 \text{ m/s} \longleftarrow V_t$$

From Eq. 12.30d we can obtain a value of A_t/A^*

$$\frac{A_t}{A^*} = \frac{1}{M_t}\left[\frac{1 + \frac{k-1}{2} M_t^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)} = \frac{1}{0.68}\left[\frac{1 + 0.2(0.68)^2}{1.2}\right]^{3.00} = 1.11$$

but at this point A^* is not known.

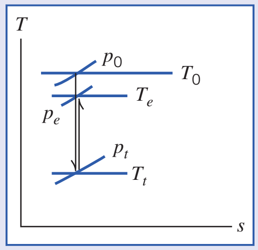
Since $M_t < 1$, flow at the exit must be subsonic. Therefore, $p_e = p_b$. Stagnation properties are constant, so

$$\frac{p_0}{p_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/(k-1)}$$

Solving for M_e gives

$$M_e = \left\{ \left[\left(\frac{p_0}{p_e} \right)^{(k-1)/k} - 1 \right] \frac{2}{k-1} \right\}^{1/2} = \left\{ \left[\left(\frac{1.0 \times 10^6}{9.54 \times 10^5} \right)^{0.286} - 1 \right] (5) \right\}^{1/2} = 0.26 \longleftarrow \qquad M_e$$

The Ts diagram for this flow is



Since A_e and M_e are known, we can compute $A*$. From Eq. 12.30d

$$\frac{A_e}{A^*} = \frac{1}{M_e} \left[\frac{1 + \frac{k-1}{2} M_e^2}{\frac{k+1}{2}} \right]^{(k+1)/2(k-1)} = \frac{1}{0.26} \left[\frac{1 + 0.2(0.26)^2}{1.2} \right]^{3.00} = 2.317$$

Thus,

$$A^* = \frac{A_e}{2.317} = \frac{0.001 \text{ m}^2}{2.317} = 4.32 \times 10^{-4} \text{ m}^2$$

and

$$A_t = 1.110 A^* = (1.110)(4.32 \times 10^{-4} \text{ m}^2)$$

$$= 4.80 \times 10^{-4} \text{ m}^2 \longleftarrow \qquad A_t$$

This problem illustrates use of the isentropic equations, Eq. 12.30a for flow in a C-D nozzle that is not choked.

- Note that use of Eq. 12.30d allowed us to obtain the throat area without needing to first compute other properties.

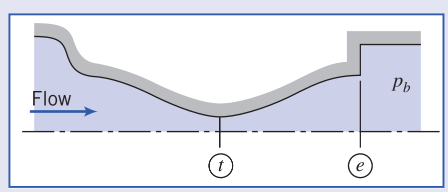
The *Excel* workbook for this example is convenient for performing the calculations (using either the isentropic equations or the basic equations). (The *Excel* add-ins for isentropic flow, on the website, also make calculations much easier.)

Example 12.11 ISENTROPIC FLOW IN A CONVERGING-DIVERGING NOZZLE: CHOKED FLOW

The nozzle of Example 12.10 has a design back pressure of 87.5 kPa (abs) but is operated at a back pressure of 50.0 kPa (abs). Assume flow within the nozzle is isentropic. Determine the exit Mach number and mass flow rate.

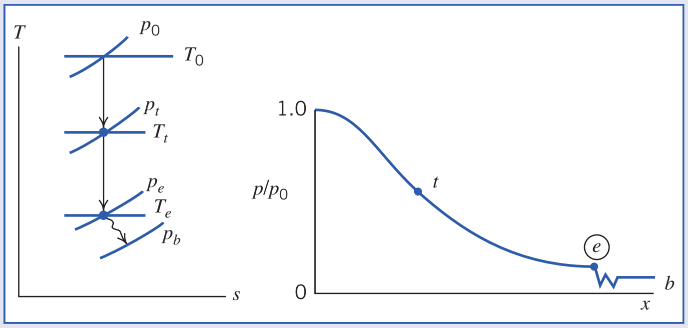
Given: Air flow through C-D nozzle as shown:

$$T_0 = 350 \text{ K}$$
$$p_0 = 1.0 \text{ MPa (abs)}$$
$$p_e(\text{design}) = 87.5 \text{ kPa (abs)}$$
$$p_b = 50.0 \text{ kPa (abs)}$$
$$A_e = 0.001 \; m^2$$
$$A_t = 4.8 \times 10^{-4} m^2 \text{ (Example 12.10)}$$



Find: (a) M_e.
(b) $\dot{m}$.

Solution: The operating back pressure is *below* the design pressure. Consequently, the nozzle is underexpanded, and the *Ts* diagram and pressure distribution will be as shown:



Flow *within* the nozzle will be isentropic, but the irreversible expansion from p_e to p_b will cause an entropy increase; $p_e = p_e(\text{design}) = 87.5$ kPa (abs).

Since stagnation properties are constant for isentropic flow, the exit Mach number can be computed from the pressure ratio. Thus

$$\frac{p_0}{p_e} = \left[1 + \frac{k-1}{2} M_e^2\right]^{k/(k-1)}$$

or

$$M_e = \left\{\left[\left(\frac{p_0}{p_e}\right)^{(k-1)/k} - 1\right]\frac{2}{k-1}\right\}^{1/2} = \left\{\left[\left(\frac{1.0\times10^6}{8.75\times10^4}\right)^{0.286} - 1\right]\frac{2}{0.4}\right\}^{1/2}$$

$$= 2.24 \longleftarrow M_e$$

Because the flow is choked, we can use Eq. 12.33b for the mass flow rate,

$$\dot{m}_{\text{choked}} = 0.04\frac{A_t p_0}{\sqrt{T_0}} \qquad (12.33b)$$

(with $\dot{m}_{\text{choked}}$ in kg/s, A_t in m^2, p_0 in Pa, and T_0 in K), so

$$\dot{m}_{\text{choked}} = 0.04 \times 4.8\times10^{-4} \times 1\times10^6/\sqrt{350}$$

$$\dot{m} = \dot{m}_{\text{choked}} = 1.04 \text{ kg/s} \longleftarrow \dot{m}$$

This problem illustrates use of the isentropic equations, Eq. 12.30a for flow in a C-D nozzle that is choked.

- Note that we used Eq. 12.33b, an "engineering equation"—that is, an equation containing a coefficient that has units. While useful here, generally these equations are no longer used in engineering because their correct use depends on using input variable values in specific units.

The *Excel* workbook for this example is convenient for performing the calculations (using either the isentropic equations or the basic equations). (The *Excel* add-ins for isentropic flow, on the website, also make calculations much easier.)

12.7 *Normal Shocks*

We mentioned normal shocks in the previous section in the context of flow through a nozzle. In practice, these irreversible discontinuities can occur in any supersonic flow field, in either internal flow or external flow. Knowledge of property changes across shocks and shock behavior is important in understanding the design of supersonic diffusers, e.g., for inlets on high-performance aircraft, and supersonic wind tunnels. Accordingly, the purpose of this section is to analyze the normal shock process.

Before applying the basic equations to normal shocks, it is important to form a clear physical picture of the shock itself. Although it is physically impossible to have discontinuities in fluid properties, the normal shock is nearly discontinuous. The thickness of a shock is about 0.2 μm or roughly 4 times the mean free path of the gas molecules [5]. Large changes in pressure, temperature, and other properties occur across this small distance. Fluid particle decelerations through the shock reach tens of millions

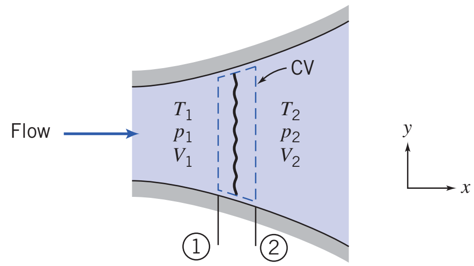


Fig. 12.14 Control volume used for analysis of normal shock.

of *g*s! These considerations justify treating the normal shock as an abrupt discontinuity; we are interested in changes occurring across the shock rather than in the details of its structure.

Consider the short control volume surrounding a normal shock standing in a passage of arbitrary shape shown in Fig. 12.14. As for isentropic flow with area variation (Section 12.6), our starting point in analyzing this normal shock is the set of basic equations (Eq. 12.24), describing one-dimensional motion that may be affected by several phenomena: area change, friction, and heat transfer. These are

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant} \tag{12.24a}$$

$$R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1 \tag{12.24b}$$

$$\frac{\delta Q}{dm} + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{12.24c}$$

$$\dot{m}(s_2 - s_1) \geq \int_{\text{CS}} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA \tag{12.24d}$$

$$p = \rho R T \tag{12.24e}$$

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1) \tag{12.24f}$$

$$\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{12.24g}$$

We recall that Equation 12.24a is *continuity*, Eq. 12.24b is a *momentum equation*, Eq. 12.24c is an *energy equation*, Eq. 12.24d is the *second law of thermodynamics*, and Eq. 12.24e, 12.24f, and 12.24g are useful *property relations* for an ideal gas with constant specific heats.

Basic Equations for a Normal Shock

We can now simplify Eqs. 12.24 for flow of an ideal gas with constant specific heats through a normal shock. The most important simplifying feature is that the width of the control volume is infinitesimal (in reality about 0.2 μm as we indicated), so $A_1 \approx A_2 \approx A$, the force due to the walls $R_x \approx 0$ because the control volume wall surface area is infinitesimal, and the heat exchange with the walls $\delta Q/dm \approx 0$, for the same reason. Hence, for this flow, our equations become

$$\rho_1 V_1 = \rho_2 V_2 = \frac{\dot{m}}{A} = \text{constant} \tag{12.34a}$$

$$p_1 A - p_2 A = \dot{m} V_2 - \dot{m} V_1$$

or, using Eq. 12.34a,

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \tag{12.34b}$$

$$h_{0_1} = h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{0_2} \tag{12.34c}$$

$$s_2 > s_1 \tag{12.34d}$$

$$p = \rho R T \tag{12.34e}$$

$$\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1) \tag{12.34f}$$

$$\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \tag{12.34g}$$

Equation 12.34 can be used to analyze flow through a normal shock. For example, if we know conditions before the shock, at section ① (i.e., p_1, ρ_1, T_1, s_1, h_1, and V_1), we can use these equations to find conditions after the shock, at section ②. We have six equations (not including the constraint of Eq. 12.34d) and six unknowns (p_2, ρ_2, T_2, s_2, h_2, and V_2). Hence, for given upstream conditions, there is a single unique downstream state. To analyze a shock, we need to solve this set of *nonlinear coupled algebraic* equations.

We can certainly use these equations for analyzing normal shocks, but we will usually find it more useful to develop normal shock functions based on M_1, the upstream Mach number. Before doing this, let us consider the set of equations. We have stated in this chapter that changes in a one-dimensional flow can be caused by area variation, friction, or heat transfer, but in deriving Eq. 12.34 we have eliminated all three causes! In this case, then, what is causing the flow to change? Perhaps there are no changes through a normal shock! Indeed, if we examine each of these equations we see that each one is satisfied—has a possible "solution"—if all properties at location ② are equal to the corresponding properties at location ① (e.g., $p_2 = p_1, T_2 = T_1$) *except* for Eq. 12.34d, which expresses the second law of thermodynamics. Nature is telling us that in the absence of area change, friction, and heat transfer, flow properties will not change *except* in a very abrupt, irreversible manner, for which the entropy increases. In fact, all properties except T_0 change through the shock. We must find a solution in which *all* of Eq. 12.34 are satisfied.

Because they are a set of nonlinear coupled equations, it is difficult to use Eq. 12.34 to see exactly what happens through a normal shock. We will postpone formal proof of the results we are about to present until a subsequent subsection, where we recast the equations in terms of the incoming Mach number. This recasting is rather mathematical, so we present results of the analysis here for clarity.

It turns out that a normal shock can occur only when the incoming flow is supersonic. Fluid flows will generally gradually adjust to downstream conditions (e.g., an obstacle in the flow) as the pressure field redirects the flow (e.g., around the object). However, if the flow is moving at such a speed that the pressure field cannot propagate upstream (when the flow speed, V, is greater than the local speed of sound, c, or in other words $M > 1$), then the fluid has to "violently" adjust to the downstream conditions. The shock that a supersonic flow may encounter is like a hammer blow that each fluid particle experiences; the pressure suddenly increases through the shock, so that, at the instant a particle is passing through the shock, there is a very large negative pressure gradient. This pressure gradient causes a dramatic reduction in speed, V, and hence a rapid rise in temperature, T, as kinetic energy is converted to internal thermal energy.

We may wonder what happens to the density because both the temperature and pressure rise through the shock, leading to opposing changes in density; it turns out that the density, ρ, increases through the shock. Because the shock is adiabatic but highly irreversible, entropy, s, increases through the shock. Finally, we see that as speed, V, decreases and the speed of sound, c, increases (because temperature, T,

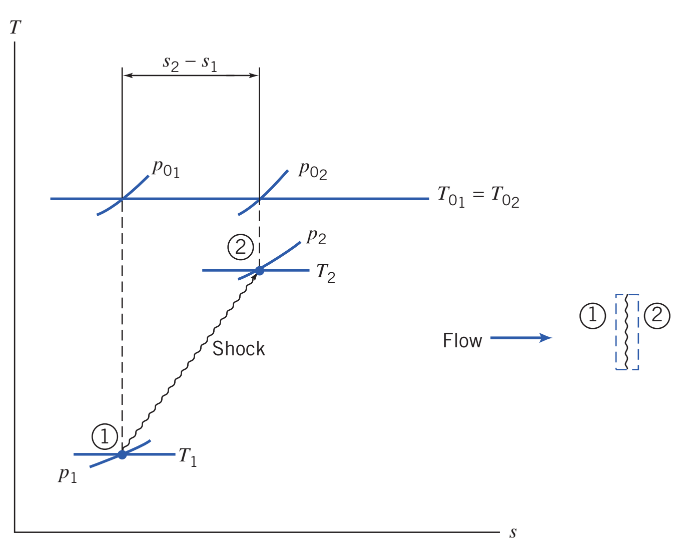


Fig. 12.15 Schematic of normal-shock process on the *Ts* plane.

Table 12.1
Summary of Property Changes Across a Normal Shock

Property	Effect	Obtained from:
Stagnation temperature	$T_0 = \text{Constant}$	Energy equation
Entropy	$s \Uparrow$	Second law
Stagnation pressure	$p_0 \Downarrow$	Ts diagram
Temperature	$T \Uparrow$	Ts diagram
Velocity	$V \Downarrow$	Energy equation, and effect on T
Density	$\rho \Uparrow$	Continuity equation, and effect on V
Pressure	$p \Uparrow$	Momentum equation, and effect on V
Mach number	$M \Downarrow$	$M = V/c$, and effects on V and T

increases) through the normal shock, the Mach number, M, decreases; in fact, we will see later that it always becomes subsonic. These results are shown graphically in Fig. 12.15 and in tabular form in Table 12.1.

Normal-Shock Flow Functions for One-Dimensional Flow of an Ideal Gas

The basic equations, Eq. 12.34, can be used to analyze flows that experience a normal shock. However, it is often more convenient to use Mach number-based equations, in this case based on the incoming Mach number, M_1. This involves three steps: First, we obtain property ratios (e.g., T_2/T_1 and p_2/p_1) in terms of M_1 and M_2, then we develop a relation between M_1 and M_2, and finally, we use this relation to obtain expressions for property ratios in terms of upstream Mach number, M_1.

The temperature ratio can be expressed as

$$\frac{T_2}{T_1} = \frac{T_2}{T_{0_2}} \frac{T_{0_2}}{T_{0_1}} \frac{T_{0_1}}{T_1}$$

Since stagnation temperature is constant across the shock, we have

$$\frac{T_2}{T_1} = \frac{1 + \dfrac{k-1}{2} M_1^2}{1 + \dfrac{k-1}{2} M_2^2} \tag{12.35}$$

A velocity ratio may be obtained by using

$$\frac{V_2}{V_1} = \frac{M_2 c_2}{M_1 c_1} = \frac{M_2 \sqrt{kRT_2}}{M_1 \sqrt{kRT_1}} = \frac{M_2}{M_1} \sqrt{\frac{T_2}{T_1}}$$

or

$$\frac{V_2}{V_1} = \frac{M_2}{M_1} \left[\frac{1 + \dfrac{k-1}{2} M_1^2}{1 + \dfrac{k-1}{2} M_2^2} \right]^{1/2}$$

A ratio of densities may be obtained from the continuity equation

$$\rho_1 V_1 = \rho_2 V_2 \tag{12.34a}$$

so that

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{M_1}{M_2} \left[\frac{1 + \dfrac{k-1}{2} M_2^2}{1 + \dfrac{k-1}{2} M_1^2} \right]^{1/2} \tag{12.36}$$

Finally, we have the momentum equation,

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2 \tag{12.34b}$$

Substituting $\rho = p/RT$, and factoring out pressures, gives

$$p_1\left[1 + \frac{V_1^2}{RT_1}\right] = p_2\left[1 + \frac{V_2^2}{RT_2}\right]$$

Since

$$\frac{V^2}{RT} = k\frac{V^2}{kRT} = kM^2$$

then

$$p_1\left[1 + kM_1^2\right] = p_2\left[1 + kM_2^2\right]$$

Finally,

$$\frac{p_2}{p_1} = \frac{1 + kM_1^2}{1 + kM_2^2} \tag{12.37}$$

To solve for M_2 in terms of M_1, we must obtain another expression for one of the property ratios given by Eqs. 12.35 through 12.37.

From the ideal gas equation of state, the temperature ratio may be written as

$$\frac{T_2}{T_1} = \frac{p_2/\rho_2 R}{p_1/\rho_1 R} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

Substituting from Eqs. 12.36 and 12.37 yields

$$\frac{T_2}{T_1} = \left[\frac{1 + kM_1^2}{1 + kM_2^2}\right]\frac{M_2}{M_1}\left[\frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2}\right]^{1/2} \tag{12.38}$$

Equations 12.35 and 12.38 are two equations for T_2/T_1. We can combine them and solve for M_2 in terms of M_1. Combining and canceling gives

$$\left[\frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2}\right]^{1/2} = \frac{M_2}{M_1}\left[\frac{1 + kM_1^2}{1 + kM_2^2}\right]$$

Squaring, we obtain

$$\frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2} = \frac{M_2^2}{M_1^2}\left[\frac{1 + 2kM_1^2 + k^2M_1^4}{1 + 2kM_2^2 + k^2M_2^4}\right]$$

which may be solved explicitly for M_2^2. Two solutions are obtained:

$$M_2^2 = M_1^2 \tag{12.39a}$$

and

$$M_2^2 = \frac{M_1^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_1^2 - 1} \tag{12.39b}$$

Obviously, the first of these is trivial. The second expresses the unique dependence of M_2 on M_1.

Now, having a relationship between M_2 and M_1, we can solve for property ratios across a shock. Knowing M_1, we obtain M_2 from Eq. 12.39b; the property ratios can be determined subsequently from Eqs. 12.35 through 12.37.

Since the stagnation temperature remains constant, the stagnation temperature ratio across the shock is unity. The ratio of stagnation pressures is evaluated as

$$\frac{p_{0_2}}{p_{0_1}} = \frac{p_{0_2}}{p_2}\frac{p_2}{p_1}\frac{p_1}{p_{0_1}} = \frac{p_2}{p_1}\left[\frac{1+\frac{k-1}{2}M_2^2}{1+\frac{k-1}{2}M_1^2}\right]^{k/(k-1)} \tag{12.40}$$

Combining Eqs. 12.37 and 12.39b, we obtain (after considerable algebra)

$$\frac{p_2}{p_1} = \frac{1+kM_1^2}{1+kM_2^2} = \frac{2k}{k+1}M_1^2 - \frac{k-1}{k+1} \tag{12.41}$$

Using Eqs. 12.39b and 12.41, we find that Eq. 12.40 becomes

$$\frac{p_{0_2}}{p_{0_1}} = \frac{\left[\dfrac{\frac{k+1}{2}M_1^2}{1+\frac{k-1}{2}M_1^2}\right]^{k/(k-1)}}{\left[\frac{2k}{k+1}M_1^2 - \frac{k-1}{k+1}\right]^{1/(k-1)}} \tag{12.42}$$

After substituting for M_2^2 from Eq.12.39b into Eqs. 12.35 and 12.36, we summarize the set of Mach number-based equations (renumbered for convenience) for use with an ideal gas passing through a normal shock:

$$M_2^2 = \frac{M_1^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_1^2 - 1} \tag{12.43a}$$

$$\frac{p_{0_2}}{p_{0_1}} = \frac{\left[\dfrac{\frac{k+1}{2}M_1^2}{1+\frac{k-1}{2}M_1^2}\right]^{k/(k-1)}}{\left[\frac{2k}{k+1}M_1^2 - \frac{k-1}{k+1}\right]^{1/(k-1)}} \tag{12.43b}$$

$$\frac{T_2}{T_1} = \frac{\left(1+\frac{k-1}{2}M_1^2\right)\left(kM_1^2 - \frac{k-1}{2}\right)}{\left(\frac{k+1}{2}\right)^2 M_1^2} \tag{12.43c}$$

$$\frac{p_2}{p_1} = \frac{2k}{k+1}M_1^2 - \frac{k-1}{k+1} \tag{12.43d}$$

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{\frac{k+1}{2}M_1^2}{1+\frac{k-1}{2}M_1^2} \tag{12.43e}$$

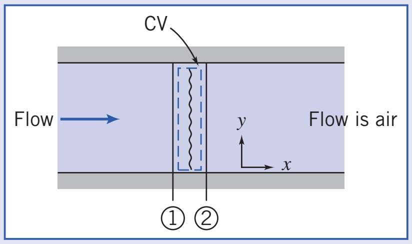
Equation 12.43 is useful for analyzing flow through a normal shock. Note that all changes through a normal shock depend only on M_1, the incoming Mach number and the fluid property, k, the ratio of specific heats. The equations are usually preferable to the original equations, Eq. 12.34, because they provide explicit, uncoupled expressions for property changes; Eqs. 12.34 are occasionally useful too. Note that Eq. 12.43d requires $M_1 > 1$ for $p_2 > p_1$, which agrees with our previous discussion. The ratio p_2/p_1 is known as the *strength* of the shock; the higher the incoming Mach number, the stronger (more violent) the shock.

Equation 12.43, while quite complex algebraically, provides explicit property relations in terms of the incoming Mach number, M_1. They are easily programmed and there are also interactive websites that make them available (see, e.g., [4]). The equations can also be programmed in *Excel* and spreadsheets are available from the website; with the add-ins, functions are available for computing M_2, and the stagnation pressure, temperature, pressure, and density/velocity ratios, from M_1, and M_1 from these ratios. Appendix D.2 lists flow functions for M_2 and property ratios p_{0_2}/p_{0_1}, T_2/T_1, p_2/p_1, and $\rho_2/\rho_1 (V_1/V_2)$ in terms of M_1 for normal-shock flow of an ideal gas. A table of values, as well as a plot of these property ratios, is presented for air ($k = 1.4$) for a limited range of Mach numbers. The associated *Excel* workbook, *Normal-Shock Relations*, can be used to print a larger table of values for air and other ideal gases. A problem involving a normal shock is solved in Example 12.12.

Example 12.12 NORMAL SHOCK IN A DUCT

A normal shock stands in a duct. The fluid is air, which may be considered an ideal gas. Properties upstream from the shock are $T_1 = 5°\text{C}$, $p_1 = 65.0$ kPa (abs), and $V_1 = 668$ m/s. Determine properties downstream and $s_2 - s_1$. Sketch the process on a Ts diagram.

Given: Normal shock in a duct as shown:



$$T_1 = 5°\text{C}$$
$$P_1 = 65.0 \text{ kPa (abs)}$$
$$V_1 = 668 \text{ m/s}$$

Find: (a) Properties at section ②.
(b) $s_2 - s_1$.
(c) Ts diagram.

Solution: First compute the remaining properties at section ①. For an ideal gas,

$$\rho_1 = \frac{p_1}{RT_1} = 6.5 \times 10^4 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287 \text{ N}\cdot\text{m}} \times \frac{1}{278 \text{ K}} = 0.815 \text{ kg/m}^3$$

$$c_1 = \sqrt{kRT_1} = \left[1.4 \times 287 \frac{\text{N}\cdot\text{m}}{\text{kg}\cdot\text{K}} \times 278 \text{ K} \times \frac{\text{kg}\cdot\text{m}}{\text{N}\cdot\text{s}^2}\right]^{1/2} = 334 \text{ m/s}$$

Then

$$M_1 = \frac{V_1}{c_1} = \frac{668}{334} = 2.00,$$ and (using isentropic stagnation relations, Eq. 12.21a and 12.21b)

$$T_{0_1} = T_1\left(1 + \frac{k-1}{2}M_1^2\right) = 278 \text{ K}[1 + 0.2(2.0)^2] = 500 \text{ K}$$

$$p_{0_1} = p_1\left(1 + \frac{k-1}{2}M_1^2\right)^{k/(k-1)} = 65.0 \text{ kPa}[1 + 0.2(2.0)^2]^{3.5} = 509 \text{ kPa (abs)}$$

From the normal-shock flow functions, Eq. 12.43, at $M_1 = 2.0$,

M_1	M_2	p_{0_2}/p_{0_1}	T_2/T_1	p_2/p_1	V_2/V_1
2.00	0.5774	0.7209	1.687	4.500	0.3750

From these data

$$T_2 = 1.687T_1 = (1.687)278 \text{ K} = 469 \text{ K} \xleftarrow{\qquad T_2 \qquad}$$

$$p_2 = 4.500p_1 = (4.500)65.0 \text{ kPa} = 293 \text{ kPa (abs)} \xleftarrow{\qquad p_2 \qquad}$$

$$V_2 = 0.3750V_1 = (0.3750)668 \text{ m/s} = 251 \text{ m/s} \xleftarrow{\qquad V_2 \qquad}$$

For an ideal gas,

$$\rho_2 = \frac{p_2}{RT_2} = 2.93 \times 10^5 \frac{\text{N}}{\text{m}^2} \times \frac{\text{kg}\cdot\text{K}}{287\ \text{N}\cdot\text{m}} \times \frac{1}{469\ \text{K}} = 2.18\ \text{kg/m}^3 \qquad \rho_2$$

Stagnation temperature is constant in adiabatic flow. Thus

$$T_{0_2} = T_{0_1} = 500\ \text{K} \qquad T_{0_2}$$

Using the property ratios for a normal shock, we obtain

$$p_{0_2} = p_{0_1}\frac{p_{0_2}}{p_{0_1}} = 509\ \text{kPa}\,(0.7209) = 367\ \text{kPa}\,(\text{abs}) \qquad p_{0_2}$$

For the change in entropy (Eq. 12.34g),

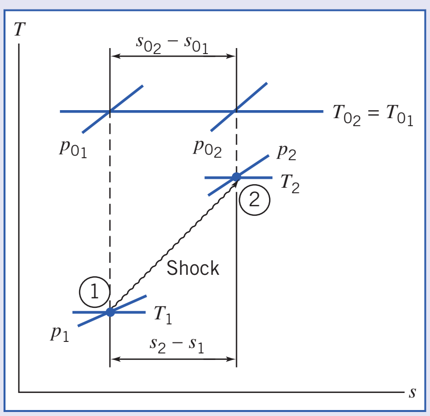
$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

But $s_{0_2} - s_{0_1} = s_2 - s_1$, so

$$s_{0_2} - s_{0_1} = s_2 - s_1 = c_p \overbrace{\ln \frac{T_{0_2}}{T_{0_1}}}^{=0} - R \ln \frac{p_{0_2}}{p_{0_1}} = -0.287 \frac{\text{kJ}}{\text{kg}\cdot\text{K}} \times \ln(0.7209)$$

$$s_2 - s_1 = 0.0939\ \text{kJ/(kg}\cdot\text{K)} \qquad s_2 - s_1$$

The Ts diagram is



This problem illustrates the use of the normal shock relations, Eq. 12.43, for analyzing flow of an ideal gas through a normal shock.

The *Excel* workbook for this problem is convenient for performing the calculations. (Alternatively, the normal shock relations *Excel* add-ins, available on the website, are useful for these calculations.)

12.8 *Supersonic Channel Flow with Shocks*

Supersonic flow is a necessary condition for a normal shock to occur and the possibility of a normal shock must be considered in any supersonic flow. Sometimes, a shock *must* occur to match a downstream pressure condition; it is desirable to determine if a shock will occur and the shock location when it does occur.

In this section, isentropic flow in a converging-diverging nozzle (Section 12.6) is extended to include shocks and complete our discussion of flow in a converging-diverging nozzle operating under varying back pressures. The pressure distribution through a nozzle for different back pressures is shown in Fig. 12.16.

Four flow regimes are possible. In Regime I the flow is subsonic throughout. The flow rate increases with decreasing back pressure. At condition (*iii*), which forms the dividing line between Regimes I and II, flow at the throat is sonic, and $M_t = 1$.

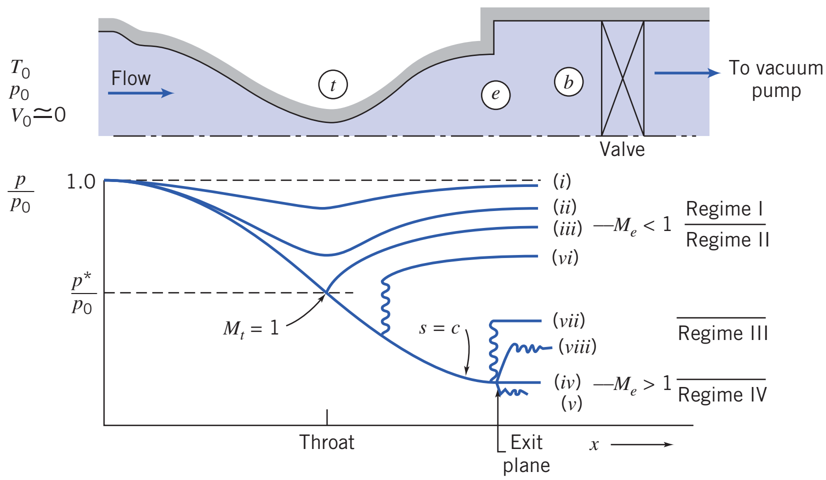


Fig. 12.16 Pressure distributions for flow in a converging-diverging nozzle for different back pressures.

As the back pressure is lowered below condition (iii), a normal shock appears downstream from the throat, as shown by condition (vi). There is a pressure rise across the shock. Since the flow is subsonic $(M<1)$ after the shock, the flow decelerates, with an accompanying increase in pressure, through the diverging channel. As the back pressure is lowered further, the shock moves downstream until it appears at the exit plane (condition *vii*). In Regime II, as in Regime I, the exit flow is subsonic, and consequently $p_e = p_b$. Since flow properties at the throat are constant for all conditions in Regime II, the flow rate in Regime II does not vary with back pressure.

In Regime III, as exemplified by condition $(viii)$, the back pressure is higher than the exit pressure, but not high enough to sustain a normal shock in the exit plane. The flow adjusts to the back pressure through a series of oblique compression shocks outside the nozzle; these oblique shocks cannot be treated by one-dimensional theory.

As previously noted in Section 12.6, condition (iv) represents the design condition. In Regime IV the flow adjusts to the lower back pressure through a series of oblique expansion waves outside the nozzle; these oblique expansion waves cannot be treated by one-dimensional theory.

The Ts diagram for converging-diverging nozzle flow with a normal shock is shown in Fig. 12.17; state ① is located immediately upstream from the shock and state ② is immediately downstream. The entropy increase across the shock moves the subsonic downstream flow to a new isentropic line. The critical temperature is constant, so p_2^* is lower than p_1^*. Since $\rho^* = p^*/RT^*$, the critical density downstream also is reduced. To carry the same mass flow rate, the downstream flow must have a larger critical area. From continuity (and the equation of state), the critical area ratio is the inverse of the critical pressure ratio, i.e., across a shock, $p^*A^* = \text{constant}$.

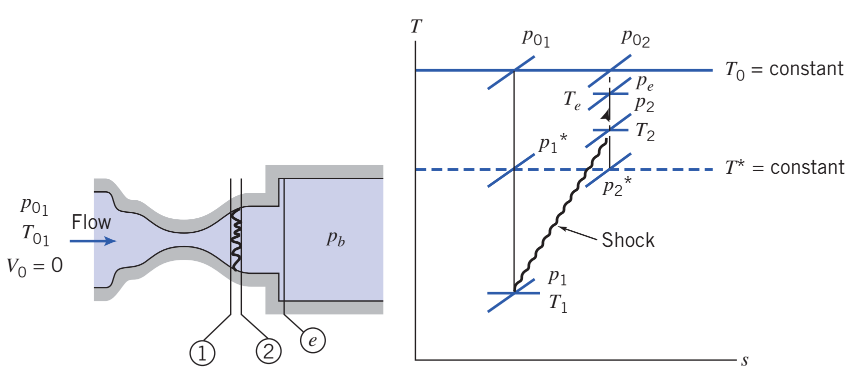


Fig. 12.17 Schematic Ts diagram for flow in a converging-diverging nozzle with a normal shock.

If the Mach number (or position) of the normal shock in the nozzle is known, the exit-plane pressure can be calculated directly. In the more realistic situation, the exit-plane pressure is specified, and the position and strength of the shock are unknown. The subsonic flow downstream must leave the nozzle at the back pressure, so $p_b = p_e$. Then

$$\frac{p_b}{p_{0_1}} = \frac{p_e}{p_{0_1}} = \frac{p_e}{p_{0_2}}\frac{p_{0_2}}{p_{0_1}} = \frac{p_e}{p_{0_2}}\frac{A_1^*}{A_2^*} = \frac{p_e}{p_{0_2}}\frac{A_t}{A_e}\frac{A_e}{A_2^*} \tag{12.44}$$

Because we have isentropic flow from state ② (after the shock) to the exit plane, $A_2^* = A_e^*$ and $p_{0_2} = p_{0_e}$. Then from Eq. 12.44 we can write

$$\frac{p_e}{p_{0_1}} = \frac{p_e}{p_{0_2}}\frac{A_t}{A_e}\frac{A_e}{A_2^*} = \frac{p_e}{p_{0_e}}\frac{A_t}{A_e}\frac{A_e}{A_e^*}$$

Rearranging,

$$\frac{p_e}{p_{0_1}}\frac{A_e}{A_t} = \frac{p_e}{p_{0_e}}\frac{A_e}{A_e^*} \tag{12.45}$$

In Eq. 12.45 the left side contains known quantities, and the right side is a function of the exit Mach number M_e only. The pressure ratio is obtained from the stagnation pressure relation (Eq. 12.21a); the area ratio is obtained from the isentropic area relation (Eq. 12.30d). Finding M_e from Eq. 12.45 usually requires iteration. The magnitude and location of the normal shock can be found once M_e is known by rearranging Eq. 12.45 (remembering that $p_{0_2} = p_{0_e}$),

$$\frac{p_{0_2}}{p_{0_1}} = \frac{A_t}{A_e}\frac{A_e}{A_e^*} \tag{12.46}$$

In Eq. 12.46 the right side is known (the first area ratio is given and the second is a function of M_e only), and the left side is a function of the Mach number before the shock, M_1, only. Hence, M_1 can be found. The area at which this shock occurs can then be found from the isentropic area relation (Eq. 12.30d, with $A^* = A_t$) for isentropic flow between the throat and state ①.

In this introductory chapter on compressible flow, we have covered some of the basic flow phenomena and presented the equations that allow us to evaluate the flow properties in some of the simpler flow situations. There are many more complex compressible flow situations, and we provide an introduction to some of these advanced topics on the website. Shock formation in a CD nozzle, one-dimensional flows with friction and/or heat transfer, and two-dimensional shock and expansion waves are covered in these sections.

12.8 *Supersonic Channel Flow with Shocks (Continued, on the Web)*

12.9 *Flow in a Constant-Area Duct with Friction (on the Web)*

12.10 *Frictionless Flow in a Constant-Area Duct with Heat Exchange (on the Web)*

12.11 *Oblique Shocks and Expansion Waves (on the Web)*

12.12 *Summary and Useful Equations*

In this chapter, we:

✓ Reviewed the basic equations used in thermodynamics, including isentropic relations.
✓ Introduced some compressible flow terminology, such as definitions of the Mach number and subsonic, supersonic, transonic, and hypersonic flows.
✓ Learned about several phenomena having to do with sound, including that the speed of sound in an ideal gas is a function of temperature only ($c = \sqrt{kRT}$) and that the Mach cone and Mach angle determine when a supersonic vehicle is heard on the ground.

- ✓ Learned that there are two useful reference states for a compressible flow: the isentropic stagnation condition, and the isentropic critical condition.
- ✓ Developed a set of governing equations (continuity, the momentum equation, the first and second laws of thermodynamics, and equations of state) for one-dimensional flow of a compressible fluid (in particular an ideal gas) as it may be affected by area change, friction, heat exchange, and normal shocks.
- ✓ Simplified these equations for isentropic flow affected only by area change and developed isentropic relations for analyzing such flows.
- ✓ Simplified the equations for flow through a normal shock, and developed normal-shock relations for analyzing such flows.

While investigating the above flows we developed insight into some interesting compressible flow phenomena, including:

- ✓ Use of Ts plots in visualizing flow behavior.
- ✓ Flow through, and necessary shape of, subsonic and supersonic nozzles and diffusers.
- ✓ The phenomenon of choked flow in converging nozzles and C-D nozzles, and the circumstances under which shock waves develop in C-D nozzles.

Note: Most of the equations in the table below have a number of constraints or limitations. *Be sure to refer to their page numbers for details*! In particular, most of them assume an ideal gas, with constant specific heats.

Useful Equations

Definition of Mach number M:	$M \equiv \frac{V}{c}$	(12.13)	Page 562
Speed of sound c:	$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right)_s}$	(12.16)	Page 565
Speed of sound c (solids and liquids):	$c = \sqrt{E_v/\rho}$	(12.17)	Page 565
Speed of sound c (ideal gas):	$c = \sqrt{kRT}$	(12.18)	Page 565
Mach cone angle α:	$\alpha = \sin^{-1}\left(\frac{1}{M}\right)$	(12.19)	Page 568
Isentropic pressure ratio (ideal gas, constant specific heats):	$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$	(12.21a)	Page 572
Isentropic temperature ratio (ideal gas, constant specific heats):	$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$	(12.21b)	Page 573
Isentropic density ratio (ideal gas, constant specific heats):	$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)}$	(12.21c)	Page 573
Critical pressure ratio (ideal gas, constant specific heats):	$\frac{p_0}{p*} = \left[\frac{k+1}{2}\right]^{k/(k-1)}$	(12.22a)	Page 576
Critical temperature ratio (ideal gas, constant specific heats):	$\frac{T_0}{T*} = \frac{k+1}{2}$	(12.22b)	Page 576
Critical density ratio (ideal gas, constant specific heats):	$\frac{\rho_0}{\rho^*} = \left[\frac{k+1}{2}\right]^{1/(k-1)}$	(12.22c)	Page 576
Critical velocity V^* (ideal gas, constant specific heats):	$V^* = c* = \sqrt{\frac{2k}{k+1}RT_0}$	(12.23)	Page 576

(Continued)

Table (*Continued*)

One-dimensional flow equations:	$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \rho V A = \dot{m} = \text{constant}$ $R_x + p_1 A_1 - p_2 A_2 = \dot{m} V_2 - \dot{m} V_1$ $\frac{\delta Q}{dm} + h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2}$ $\dot{m}(s_2 - s_1) \geq \int_{CS} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA$ $p = \rho R T$ $\Delta h = h_2 - h_1 = c_p \Delta T = c_p (T_2 - T_1)$ $\Delta s = s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$	(12.24a) (12.24b) (12.24c) (12.24d) (12.24e) (12.24f) (12.24g)	Page 577 Page 579
Isentropic relations: [Note: These equations are a little cumbersome for practical use by hand. They are listed (and tabulated and plotted for air) in Appendix D. The *Excel* add-ins from the website are useful for computing with these equations.]	$\frac{p_0}{p} = f(M)$ $\frac{T_0}{T} = f(M)$ $\frac{\rho_0}{\rho} = f(M)$ $\frac{A}{A^*} = f(M)$	(12.30a) (12.30b) (12.30c) (12.30d)	Page 583 Page 583 Page 584
Pressure ratio for choked converging nozzle, $p_e/p_0\vert_{\text{choked}}$:	$\frac{p_e}{p_0}\Big\vert_{\text{choked}} = \frac{p*}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)}$	(12.31)	Page 589
Mass flow rate for choked converging nozzle:	$\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)}$	(12.32a)	Page 589
Mass flow rate for choked converging nozzle (SI units):	$\dot{m}_{\text{choked}} = 0.04 \frac{A_e p_0}{\sqrt{T_0}}$	(12.32b)	Page 589
Mass flow rate for choked converging-diverging nozzle:	$\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}}\left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)}$	(12.33a)	Page 593
Mass flow rate for choked converging-diverging nozzle (SI units):	$\dot{m}_{\text{choked}} = 0.04 \frac{A_t p_0}{\sqrt{T_0}}$	(12.33b)	Page 593
Normal shock relations: [Note: These equations are too cumbersome for practical use by hand. They are listed (and tabulated and plotted for air) in Appendix D. The *Excel* add-ins from the website are useful for computing with these equations.]	$M_2 = f(M_1)$ $\frac{p_{0_2}}{p_{0_1}} = f(M_1)$ $\frac{T_2}{T_1} = f(M_1)$ $\frac{p_2}{p_1} = f(M_1)$ $\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = f(M_1)$	(12.43a) (12.43b) (12.43c) (12.43d) (12.43e)	Page 602
Useful relations for determining the normal shock location in converging-diverging nozzle:	$\frac{p_e}{p_{0_1}} \frac{A_e}{A_t} = \frac{p_e}{p_{0_e}} \frac{A_e}{A_e^*}$ $\frac{p_{0_2}}{p_{0_1}} = \frac{A_t}{A_e} \frac{A_e}{A_e^*}$	(12.45) (12.46)	Page 606 Page 606

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PROBLEMS

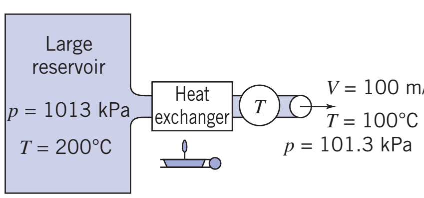
Review of Thermodynamics

12.1 Air is expanded in a steady flow process through a turbine. Initial conditions are 1300°C and 2.0 MPa absolute. Final conditions are 500°C and atmospheric pressure. Show this process on a *Ts* diagram. Evaluate the changes in internal energy, enthalpy, and specific entropy for this process.

12.2 Five kilograms of air is cooled in a closed tank from 250 to 50°C. The initial absolute pressure is 3 MPa. Compute the changes in entropy, internal energy, and enthalpy. Show the process state points on a *Ts* diagram.

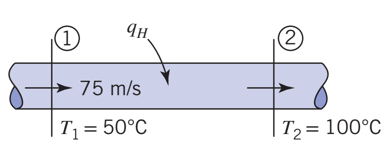
12.3 Air is contained in a piston-cylinder device. The temperature of the air is 100°C. Using the fact that for a reversible process the heat transfer $q = \int T ds$, compare the amount of heat (J/kg) required to raise the temperature of the air to 1200°C at (a) constant pressure and (b) constant volume. Verify your results using the first law of thermodynamics. Plot the processes on a *Ts* diagram.

12.4 Calculate the power delivered by the turbine per unit mass of airflow when the heat transfer in the heat exchanger is zero. Then, how does the power depend on the heat transfer through the exchanger if all other conditions remain the same? Assume that air is a perfect gas.



P12.4

12.5 If hydrogen flows as a perfect gas without friction between stations ① and ② while $q_H = 7.5 \times 10^5$ J/kg, find V_2.



P12.5

12.6 A 1-m^3 tank contains air at 0.1 MPa absolute and 20°C. The tank is pressurized to 2 MPa. Assuming that the tank is filled adiabatically and reversibly, calculate the final temperature of the air in the tank. Now assuming that the tank is filled isothermally, how much heat is lost by the air in the tank during filling? Which process (adiabatic or isothermal) results in a greater mass of air in the tank?

12.7 Air enters a turbine in steady flow at 0.5 kg/s with negligible velocity. Inlet conditions are 1300°C and 2.0 MPa absolute. The air is expanded through the turbine to atmospheric pressure. If the actual temperature and velocity at the turbine exit are 500°C and 200 m/s, determine the power produced by the turbine. Label state points on a *Ts* diagram for this process.

12.8 Natural gas, with the thermodynamic properties of methane, flows in an underground pipeline of 0.6 m diameter. The gage pressure at the inlet to a compressor station is 0.5 MPa; outlet pressure is 8.0 MPa gage. The gas temperature and speed at inlet are 13°C and 32 m/s, respectively. The compressor efficiency is $\eta = 0.85$. Calculate the mass flow rate of natural gas through the pipeline. Label state points on a *Ts* diagram for compressor inlet and outlet. Evaluate the gas temperature and speed at the compressor outlet and the power required to drive the compressor.

12.9 Carbon dioxide flows at a speed of 10 m/s in a pipe and then through a nozzle where the velocity is 50 m/s, What is the change in gas temperature between pipe and nozzle? Assume that this is an adiabatic flow of a perfect gas.

Propagation of Sound Waves

12.10 Calculate the speed of sound at 20°C for (a) hydrogen, (b) helium, (c) methane, (d) nitrogen, and (e) carbon dioxide.

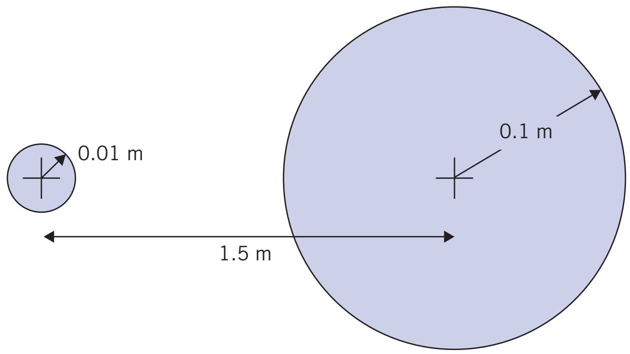
12.11 An airplane flies at 550 km/h at 1500 m altitude on a standard day. The plane climbs to 15,000 m and flies at 1200 km/h. Calculate the Mach number of flight in both cases.

12.12 For a speed of sound in steel of 4300 m/s, determine the bulk modulus of elasticity. Compare the modulus of elasticity of steel to that of water. Determine the speed of sound in steel, water, and air at atmospheric conditions. Comment on the differences.

12.13 Investigate the effect of altitude on Mach number by plotting the Mach number of a 500-mph airplane as it flies at altitudes ranging from sea level to 10 km.

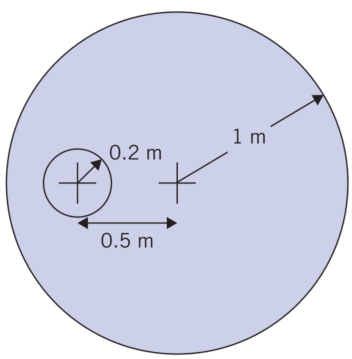
12.14 Use data for specific volume to calculate and plot the speed of sound in saturated liquid water over the temperature range from 0 to 200°C.

12.15 An object traveling in atmospheric air emits two pressure waves at different times. At an instant in time, the waves appear as in the figure. Determine the velocity and Mach number of the object and its current location.



P12.15

12.16 An object traveling in atmospheric air emits two pressure waves at different times. At an instant in time, the waves appear as in the figure. Determine the velocity and Mach number of the object and its current location.



P12.16

12.17 The temperature varies linearly from sea level to approximately 11 km altitude in the standard atmosphere. Evaluate the *lapse rate*—the rate of decrease of temperature with altitude—in the standard atmosphere. Derive an expression for the rate of change of sonic speed with altitude in an ideal gas under standard atmospheric conditions. Evaluate and plot the sonic speed from sea level to 10 km altitude.

12.18 A photograph of a bullet shows a Mach angle of 32°. Determine the speed of the bullet for standard air.

12.19 An F-4 aircraft makes a high-speed pass over an airfield on a day when $T = 35°C$. The aircraft flies at $M = 1.4$ and 200 m altitude. Calculate the speed of the aircraft. How long after it passes directly over point A on the ground does its Mach cone pass over point A?

12.20 An aircraft passes overhead at 3 km altitude. The aircraft flies at $M = 1.5$. Assume that the air temperature is constant at 20°C. Find the air speed of the aircraft. A headwind blows at 30 m/s. How long after the aircraft passes directly overhead does its sound reach a point on the ground?

12.21 A supersonic aircraft flies at 3 km altitude at a speed of 1000 m/s on a standard day. How long after passing directly above a ground observer is the sound of the aircraft heard by the ground observer?

12.22 For the conditions of Problem 12.21, find the location at which the sound wave that first reaches the ground observer was emitted.

12.23 The Concorde supersonic transport cruised at $M = 2.2$ at 17 km altitude on a standard day. How long after the aircraft passed directly above a ground observer was the sound of the aircraft heard?

Reference State: Local Isentropic Stagnation Properties

12.24 Plot the percentage discrepancy between the density at the stagnation point and the density at a location where the Mach number is M, of a compressible flow, for Mach numbers ranging from 0.05 to 0.95. Find the Mach numbers at which the discrepancy is 1 percent, 5 percent, and 10 percent.

12.25 Compute the air density in the undisturbed air and at the stagnation point of an aircraft flying at 250 m/s in air at 28 kPa and 250°C. What is the percentage increase in density? Can we approximate this as an incompressible flow?

12.26 Carbon dioxide flows in a duct at a velocity of 90 m/s, absolute pressure 140 kPa, and temperature 90°C. Calculate pressure and temperature on the nose of a small object placed in this flow.

12.27 If nitrogen at 15°C is flowing and the stagnation temperature on the nose of a small object in the flow is measured as 38°C, what is the velocity in the pipe?

12.28 An aircraft cruises at $M = 0.65$ at 10 km altitude on a standard day. The aircraft speed is deduced from measurement of the difference between the stagnation and static pressures. What is the value of this difference? Compute the air speed from this actual difference assuming (a) compressibility and (b) incompressibility. Is the discrepancy in air-speed computations significant in this case?

12.29 High-speed aircraft use "air data computers" to compute air speed from measurement of the difference between the stagnation and static pressures. Plot, as a function of actual Mach number M, for $M = 0.1$ to $M = 0.9$, the percentage error in computing the Mach number assuming incompressibility (i.e., using the Bernoulli equation), from this pressure difference. Plot the percentage error in speed, as a function of speed, of an aircraft cruising at 12 km altitude, for a range of speeds corresponding to the actual Mach number ranging from $M = 0.1$ to $M = 0.9$.

12.30 A supersonic wind tunnel test section is designed to have $M = 2.5$ at 15°C and 35 kPa absolute. The fluid is air. Determine the required inlet stagnation conditions, T_0 and p_0. Calculate the required mass flow rate for a test section area of 0.175 m^2.

12.31 What is the pressure on the nose of a bullet moving through standard sea ievel air at 300 m/s assuming that (a) the flow is incompressible and (b) the flow is compressible? Compare results.

12.32 Air flows steadily through an insulated constant area duct, where 1 denotes the inlet and 2 the outlet. Properties change along the duct as a result of friction.

(a) Beginning with the control volume form of the first law of thermodynamics, show that the equation can be reduced to

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = \text{constant}$$

(b) Denoting the constant by h_0 (the stagnation enthalpy), show that for adiabatic flow of an ideal gas with friction

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

(c) For this flow does $T_{0_1} = T_{0_2}$? $p_{0_1} = p_{0_2}$? Explain these results.

12.33 Air flows in an insulated duct. At point ①, the conditions are $M_1 = 0.1, T_1 = -20°\text{C}$ and $p_1 = 1.0$ MPa absolute. Downstream, at point ②, because of friction the conditions are $M_2 = 0.7$, $T_2 = -5.62°\text{C}$, and $p_2 = 136.5$ kPa absolute. (Four significant figures are given to minimize roundoff errors.) Compare the stagnation temperatures at points ① and ② and explain the result. Compute the stagnation pressures at points ① and ②. Can you explain how it can be that the velocity *increases* for this frictional flow? Should this process be isentropic or not? Justify your answer by computing the change in entropy between points ① and ②. Plot static and stagnation state points on a Ts diagram.

12.34 Consider steady, adiabatic flow of air through a long straight pipe with $A = 0.05\ \text{m}^2$. At the inlet section ①, the air is at 200 kPa absolute, 60°C, and 146 m/s. Downstream at section ②, the air is at 95.6 kPa absolute and 280 m/s. Determine p_{0_1}, p_{0_2}, T_{0_1}, T_{0_2}, and the entropy change for the flow. Show static and stagnation state points on a Ts diagram.

12.35 Air passes through a normal shock in a supersonic wind tunnel. Upstream conditions are $M_1 = 1.8, T_1 = 270\text{K}$, and $p_1 = 10.0$ kPa absolute. Downstream conditions are $M_2 = 0.6165, T_2 = 413.6$ K, and $p_2 = 36.13$ kPa absolute. (Four significant figures are given to minimize roundoff errors.) Evaluate local isentropic stagnation conditions (a) upstream from, and (b) downstream from, the normal shock. Calculate the change in specific entropy across the shock. Plot static and stagnation state points on a Ts diagram.

12.36 A Boeing 747 cruises at $M = 0.87$ at an altitude of 13 km on a standard day. A window in the cockpit is located where the external flow Mach number is 0.2 relative to the plane surface. The cabin is pressurized to an equivalent altitude of 2500 m in a standard atmosphere. Estimate the pressure difference across the window. Be sure to specify the direction of the net pressure force.

Critical Conditions

12.37 A CO_2 cartridge is used to propel a toy rocket. Gas in the cartridge is pressurized to 45 MPa gage and is at 25°C. Calculate the critical conditions (temperature, pressure, and flow speed) that correspond to these stagnation conditions.

12.38 Air flows from the atmosphere into an evacuated tank through a convergent nozzle of 38-mm tip diameter. If atmospheric pressure and temperature are 101.3 kPa and 15°C, respectively, what vacuum must be maintained in the tank to produce sonic velocity in the jet? What is the flow rate? What is the flow rate when the vacuum is 254 mm of mercury?

12.39 Oxygen discharges from a tank through a convergent nozzle. The temperature and velocity in the jet are −20°C and 270 m/s, respectively. What is the temperature in the tank? What is the temperature on the nose of a small object in the jet?

12.40 The hot gas stream at the turbine inlet of a JT9-D jet engine is at 1500°C, 140 kPa absolute, and $M = 0.32$. Calculate the critical conditions (temperature, pressure, and flow speed) that correspond to these conditions. Assume the fluid properties of pure air.

12.41 Carbon dioxide discharges from a tank through a convergent nozzle into the atmosphere. If the tank temperature and gage pressure are 38°C and 140 kPa, respectively, what jet temperature, pressure, and velocity can be expected? Barometric pressure is 101.3 kPa.

12.42 Steam flows steadily and isentropically through a nozzle. At an upstream section where the speed is negligible, the temperature and pressure are 450°C and 6 MPa absolute. At a section where the nozzle diameter is 2 cm, the steam pressure is 2 MPa absolute. Determine the speed and Mach number at this section and the mass flow rate of steam. Sketch the passage shape.

12.43 Nitrogen flows through a diverging section of duct with $A_1 = 0.15\ \text{m}^2$ and $A_2 = 0.45\ \text{m}^2$. If $M_1 = 0.7$ and $p_1 = 450$ kPa, find M_2 and p_2.

Isentropic Flow—Area Variation

12.44 In a given duct flow M = 2.0; the velocity undergoes a 20 percent decrease. What percent change in area was needed to accomplish this? What would be the answer if M = 0.5?

12.45 Air flows isentropically through a converging-diverging nozzle from a large tank containing air at 250°C. At two locations where the area is 1 cm², the static pressures are 200 kPa and 50 kPa. Find the mass flow rate, the throat area, and the Mach numbers at the two locations.

12.46 Air, at an absolute pressure of 60.0 kPa and 27°C, enters a passage at 486 m/s, where $A = 0.02\ \text{m}^2$. At section ① downstream, $p = 78.8$ kPa absolute. Assuming isentropic flow, calculate the Mach number at section ②. Sketch the flow passage.

12.47 Carbon dioxide flows from a tank through a convergent-divergent nozzle of 25-mm throat and 50-mm exit diameter. The absolute pressure and temperature in the tank are 241.5 kPa and 37.8°C, respectively. Calculate the mass flow rate when the absolute exit pressure is (a) 172.5 kPa and (b) 221 kPa.

12.48 A convergent-divergent nozzle of 50-mm tip diameter discharges to the atmosphere (103.2 kPa) from a tank in which air is maintained at an absolute pressure and temperature of 690 kPa and 37.8°C, respectively. What is the maximum mass flow rate that can occur through this nozzle? What throat diameter must be provided to produce this mass flow rate?

12.49 Air flows adiabatically through a duct. At the entrance, the static temperature and pressure are 310 K and 200 kPa, respectively. At the exit, the static and stagnation temperatures are 294 K and 316 K, respectively, and the static pressure is 125 kPa. Find (a) the Mach numbers of the flow at the entrance and exit and (b) the area ratio A_2/A_1.

12.50 Air flows isentropically through a converging nozzle into a receiver where the pressure is 250 kPa absolute. If the pressure is 350 kPa absolute and the speed is 150 m/s at the nozzle location where the Mach number is 0.5, determine the pressure, speed, and Mach number at the nozzle throat.

12.51 Atmospheric air at 98.5 kPa and 20°C is drawn into a vacuum tank through a convergent-divergent nozzle of 50-mm throat diameter and 75-mm exit diameter. Caiculate the largest mass flow rate that can be drawn through this nozzle under these conditions.

12.52 The exit section of a convergent-divergent nozzle is to be used for the test section of a supersonic wind tunnel. If the absolute pressure in the test section is to be 140 kPa, what pressure is required in the reservoir to produce a Mach number of 5 in the test section? For the air temperature to be $-20°C$ in the test section, what temperature is required in the reservoir? What ratio of throat area to test section area is required to meet these conditions?

12.53 Air flowing isentropically through a converging nozzle discharges to the atmosphere. At the section where the absolute pressure is 250 kPa, the temperature is 20°C and the air speed is 200 m/s. Determine the nozzle throat pressure.

12.54 Air flows from a large tank at $p = 650$ kPa absolute, $T = 550°C$ through a converging nozzle, with a throat area of 600 mm^2, and discharges to the atmosphere. Determine the mass rate of flow for isentropic flow through the nozzle.

12.55 A converging nozzle is connected to a large tank that contains compressed air at 15°C. The nozzle exit area is 0.001 m^2. The exhaust is discharged to the atmosphere. To obtain a satisfactory shadow photograph of the flow pattern leaving the nozzle exit, the pressure in the exit plane must be greater than 325 kPa gage. What pressure is required in the tank? What mass flow rate of air must be supplied if the system is to run continuously? Show static and stagnation state points on a *Ts* diagram.

12.56 Air at 0°C is contained in a large tank on the space shuttle. A converging section with exit area 1×10^{-3} m^2 is attached to the tank, through which the air exits to space at a rate of 2 kg/s. What are the pressure in the tank, and the pressure, temperature, and speed at the exit?

12.57 A large tank initially is evacuated to -10 kPa gage. Ambient conditions are 101 kPa at 20°C. At $t = 0$, an orifice of 5 mm diameter is opened in the tank wall; the vena contracta area is 65 percent of the geometric area. Calculate the mass flow rate at which air initially enters the tank. Show the process on a *Ts* diagram. Make a schematic plot of mass flow rate as a function of time. Explain why the plot is nonlinear.

12.58 Air flows isentropically through a converging nozzle attached to a large tank, where the absolute pressure is 171 kPa and the temperature is 27°C. At the inlet section, the Mach number is 0.2. The nozzle discharges to the atmosphere; the discharge area is 0.015 m^2. Determine the magnitude and direction of the force that must be applied to hold the nozzle in place.

12.59 Air enters a converging-diverging nozzle at 2 MPa absolute and 313 K. At the exit of the nozzle, the pressure is 200 kPa absolute. Assume adiabatic, frictionless flow through the nozzle. The throat area is 20 cm^2. What is the area at the nozzle exit? What is the mass flow rate of the air?

12.60 A jet transport aircraft, with pressurized cabin, cruises at 11 km altitude. The cabin temperature and pressure initially are at 25°C and equivalent to 2.5 km altitude. The interior volume of the cabin is 25 m^3. Air escapes through a small hole with effective flow area of 0.002 m^2. Calculate the time required for the cabin pressure to decrease by 40 percent. Plot the cabin pressure as a function of time.

12.61 Air, at a stagnation pressure of 7.20 MPa absolute and a stagnation temperature of 1100 K, flows isentropically through a converging-diverging nozzle having a throat area of 0.01 m^2. Determine the speed and the mass flow rate at the downstream section where the Mach number is 4.0.

12.62 A small rocket motor, fueled with hydrogen and oxygen, is tested on a thrust stand at a simulated altitude of 10 km. The motor is operated at chamber stagnation conditions of 1500 K and 8.0 MPa gage. The combustion product is water vapor, which may be treated as an ideal gas. Expansion occurs through a converging-diverging nozzle with design Mach number of 3.5 and exit area of 700 mm^2. Evaluate the pressure at the nozzle exit plane. Calculate the mass flow rate of exhaust gas. Determine the force exerted by the rocket motor on the thrust stand.

Normal Shocks

12.63 Testing of a demolition explosion is to be evaluated. Sensors indicate that the shock wave generated at the instant of explosion is 30 MPa absolute. If the explosion occurs in air at 20°C and 101 kPa, find the speed of the shock wave, and the temperature and speed of the air just after the shock passes. As an approximation assume $k = 1.4$. Why is this an approximation?

12.64 Air discharges through a convergent-divergent nozzle which is attached to a large reservoir. At a point in the nozzle a normal shock wave is detected across which the absolute pressure jumps from 69 to 207 kPa. Calculate the pressures in the throat of the nozzle and in the reservoir.

12.65 A normal shock wave exists in an airflow. The absolute pressure, velocity, and temperature just upstream from the wave are 207 kPa, 610 m/s, and $-17.8°C$, respectively. Calculate the pressure, velocity, temperature, and sonic velocity just downstream from the shock wave.

12.66 Air approaches a normal shock at $V_1 = 900$ m/s, $p_1 = 50$ kPa absolute, and $T_1 = 220$ K. What are the velocity and pressure after the shock? What would the velocity and pressure be if the flow were decelerated isentropically to the same Mach number?

12.67 Air undergoes a normal shock. Upstream, $T_1 = 35°C$, $p_1 = 229$ kPa absolute, and $V_1 = 704$ m/s. Determine the temperature and stagnation pressure of the air stream leaving the shock.

12.68 If, through a normal shock wave in air, the absolute pressure rises from 275 to 410 kPa and the velocity diminishes from 460 to 346 m/s, what temperatures are to be expected upstream and downstream from the wave?

12.69 The stagnation temperature in an airflow is 149°C upstream and downstream from a normal shock wave. The absolute stagnation pressure downstream from the shock wave is 229.5 kPa. Through the wave, the absolute pressure rises from 103.4 to 138 kPa. Determine the velocities upstream and downstream from the wave.

12.70 A supersonic aircraft cruises at $M = 2.2$ at 12 km altitude. A pitot tube is used to sense pressure for calculating air speed. A normal shock stands in front of the tube. Evaluate the local isentropic stagnation conditions in front of the shock. Estimate the stagnation pressure sensed by the pitot tube. Show static and stagnation state points and the process path on a *Ts* diagram.

12.71 The Concorde supersonic transport flew at $M = 2.2$ at 20 km altitude. Air is decelerated isentropically by the engine inlet system to a local Mach number of 1.3. The air passed through a normal shock and was decelerated further to $M = 0.4$ at the engine compressor section. Assume, as a first approximation, that this subsonic diffusion process was isentropic and use standard atmosphere data for freestream conditions. Determine the temperature, pressure, and stagnation pressure of the air entering the engine compressor.