Sichuan University - Pittsburgh Institute

ME 1071: Applied Fluids

Lecture 1 Internal Incompressible Viscous Flow

Spring 2021

Outlines

- Introduction
- Internal Flow Characteristics
- Fully Developed Internal Flows
  - Moving Plates without Pressure Gradient (Couette Flow)
  - Parallel Stationary Plates with Pressure Gradient (Poiseuille Flow)
  - Moving Plates with Pressure Gradient
  - Pipe Flow

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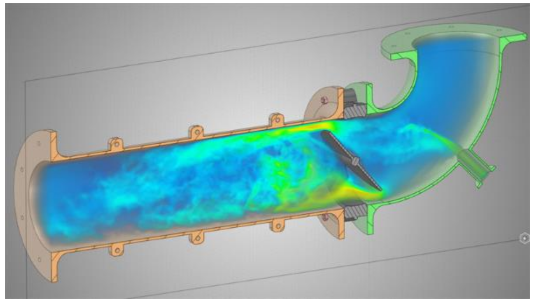
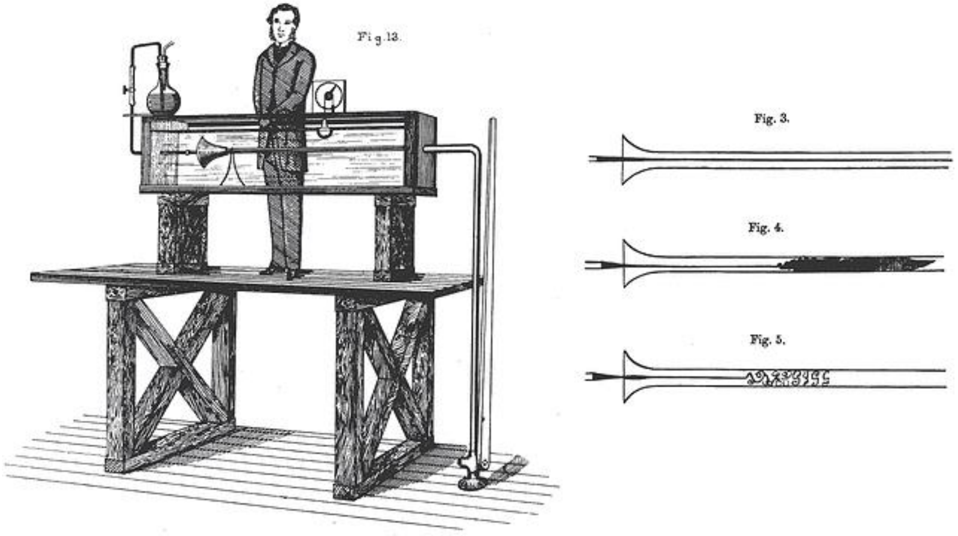
Introduction

Internal flow

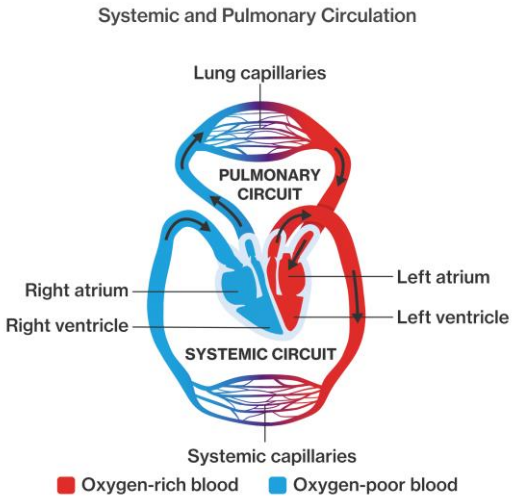
- Flows completely bounded by solid surfaces.
- Laminar or turbulent
- Analytical solutions are possible for some laminar case.



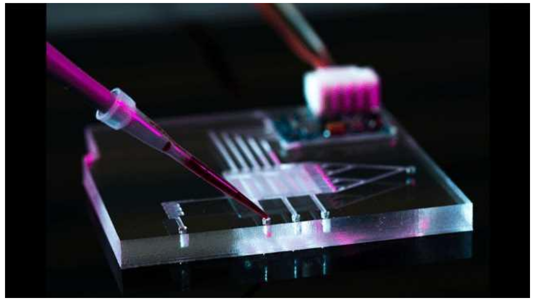
Water Supply Control



The Reynolds Experiment (1883)



Blood Circulation



Lab-on-a-chip

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Introduction

Internal flow Characteristics

- Laminar versus turbulent
- Transition to Turbulence at $\mathrm{Re} \approx 2300$
- The entrance region

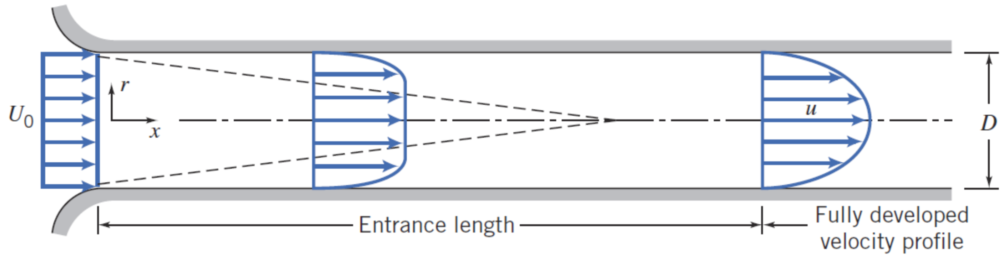
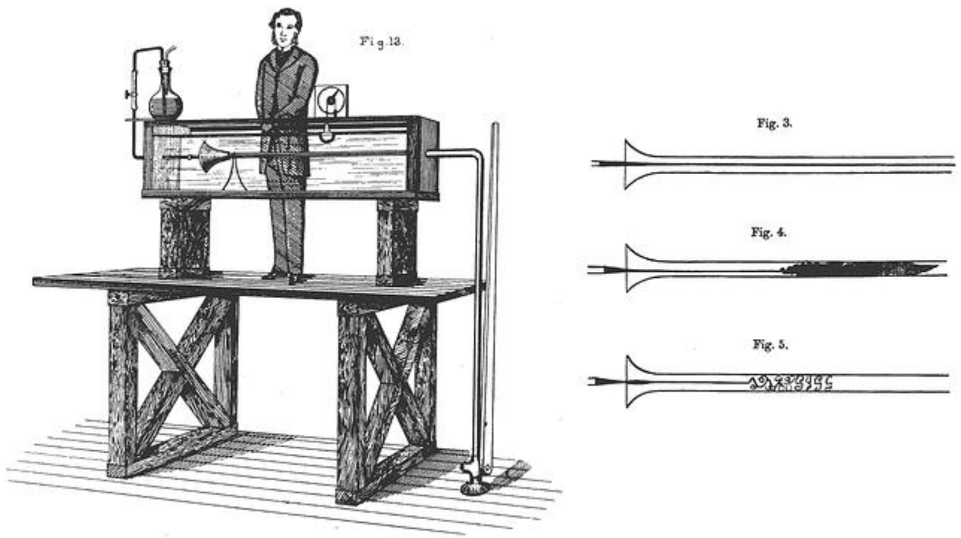


Fig. 8.1 Flow in the entrance region of a pipe.



The Reynolds Experiment (1883)

Fully developed flow

- When the velocity or temperature profile shapes no longer change with increasing distance x
- For laminar flow, the entrance length is a function of Reynolds number

$$\frac{L}{D} \simeq 0.06\,\mathrm{Re} = 0.06\,\frac{\rho \bar{V} D}{\mu}, \qquad L < 138D$$

- For turbulent flow,

$$\frac{L}{D} \simeq 25 - 40 \text{ for velocity}, \quad \frac{L}{D} > 80 \text{ for detailed motion}$$

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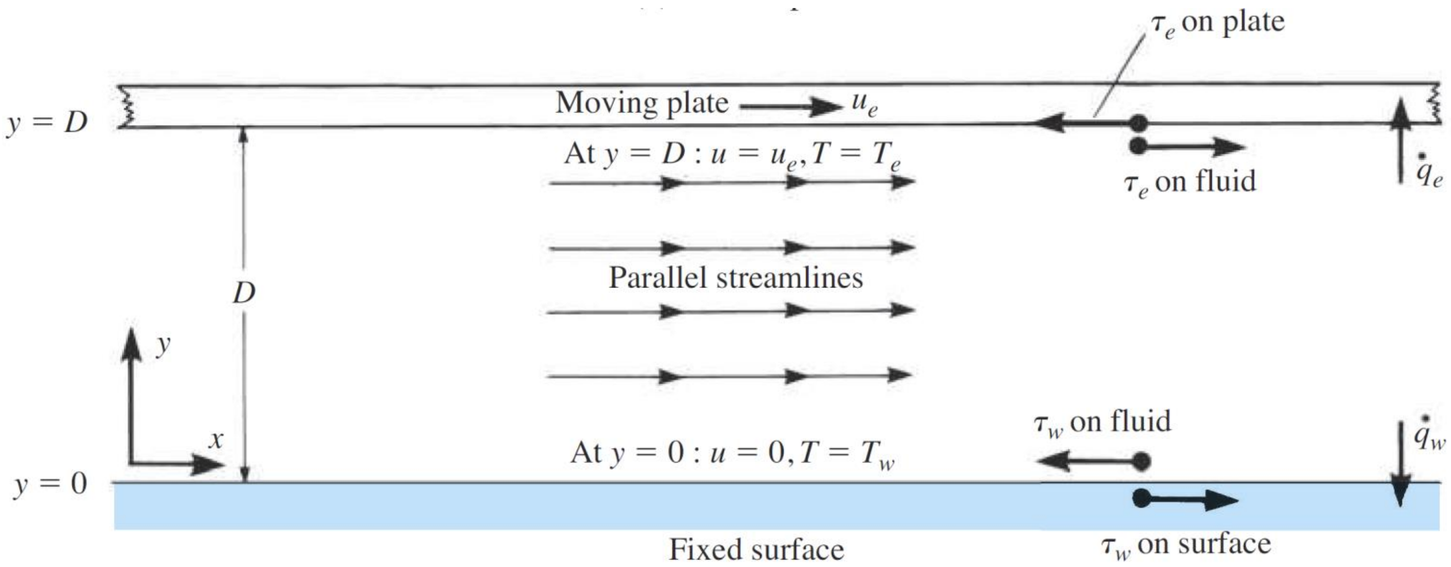
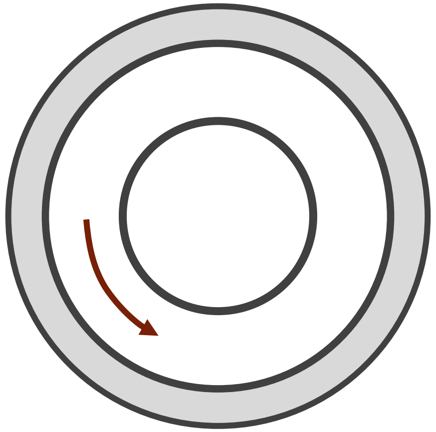
Introduction to Couette Flow

- **Couette Flow**
  - The flow of a viscous fluid in the space between two (infinite) surfaces, one of which is moving tangentially relative to the other. No pressure gradient is involved.
  - The flow is driven by the viscous drag force and the streamlines are parallel.





Model for Couette flow between two plate or two concentric cylinder (同心圆柱).

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Introduction to Couette Flow

- **General Governing Equations (Review)**

Continuity

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

Momentum

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}$$

$$\rho\frac{Dv}{Dt}=-\frac{\partial p}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt}=-\frac{\partial p}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}$$

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\qquad \tau_{xx}=\lambda(\nabla\cdot\mathbf{V})+2\mu\frac{\partial u}{\partial x}$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\qquad \tau_{yy}=\lambda(\nabla\cdot\mathbf{V})+2\mu\frac{\partial v}{\partial y}$$

$$\tau_{zx}=\tau_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\qquad \tau_{zz}=\lambda(\nabla\cdot\mathbf{V})+2\mu\frac{\partial w}{\partial z}$$

Energy

$$\rho\frac{D(e+V^2/2)}{Dt}=\rho\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$
$$-\nabla\cdot p\mathbf{V}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}$$
$$+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}$$

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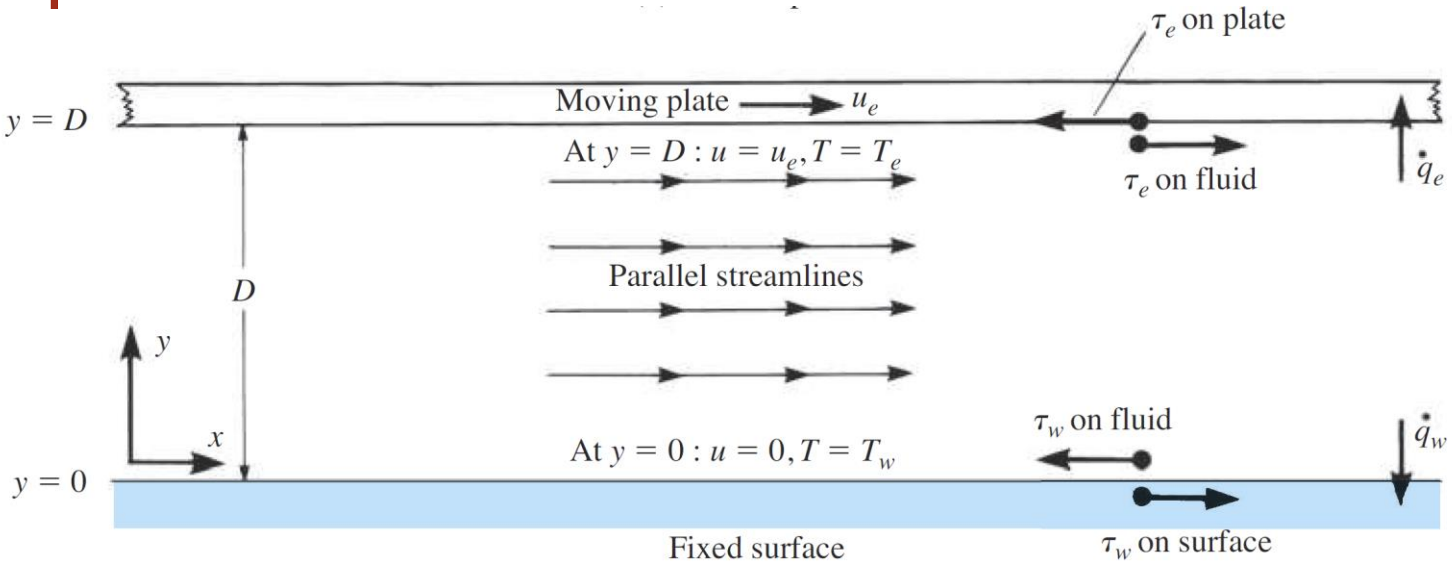
Introduction to Couette Flow

- **The Governing Equations for Couette Flow**



$$v = w = 0 \qquad \frac{\partial T}{\partial x} = \frac{\partial p}{\partial x} = 0$$

Continuity

$$\frac{\partial(\rho u)}{\partial x} = 0$$

Momentum

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial p}{\partial y} = 0$$

(pressure is constant throughout the entire flow field)

Energy

$$\frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u \frac{\partial u}{\partial y}\right) = 0$$

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Incompressible Couette Flow

- **Assumptions**
  - ρ, μ, k are treated as constants.
  - T is also constant.

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)=0$$

μ = const.

$$\frac{\partial^2 u}{\partial y^2}=0$$

$$u=ay+b$$

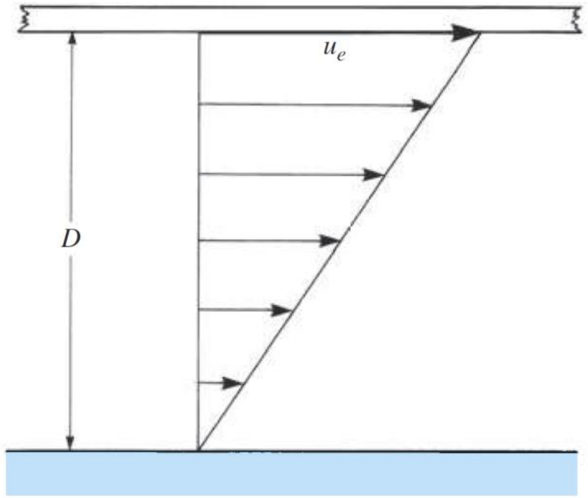
y = 0, u = 0 → b = 0;

y = D, u = u_e → a = u_e/D.

$$\boxed{u=u_e(y/D)}$$

$$\tau=\mu\frac{\partial u}{\partial y}$$

$$\tau=\mu\left(\frac{u_e}{D}\right)$$



Velocity profile for incompressible Couette flow

For incompressible Couette flow

- Velocity varies linearly across the flow.
- τ increases linearly with u_e.
- τ is inversely proportional to D.

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Fully Developed Internal Flows

Parallel Stationary Plates with Pressure Gradient (Poiseuille Flow)

- Both plates are stationary.
- The fluid is driven by pressure gradient.
- Flow is laminar and fully developed, body force is neglected.
- Boundary condition $\quad$ at $y=0 \quad u=0$; at $y=a \quad u=0$

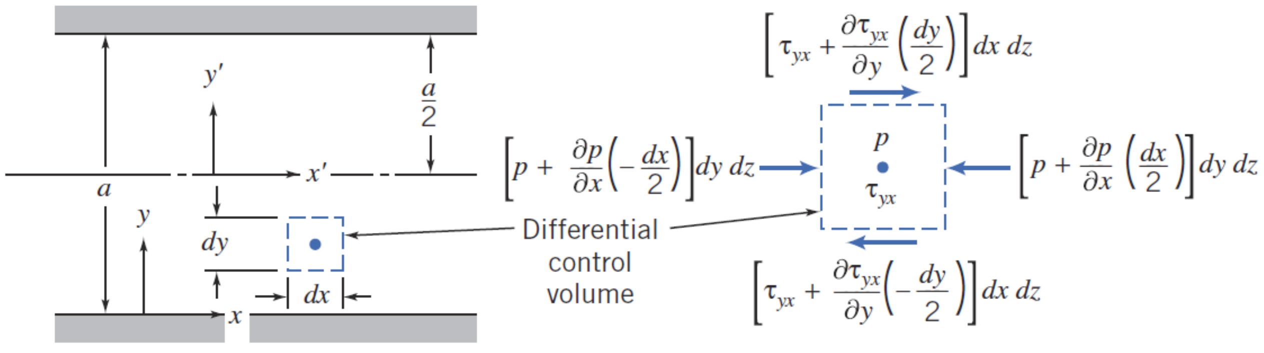
$$dF_L = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz + dF_T = \left(\tau_{yx} + \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dx\, dz$$

$$=$$

$$dF_R = -\left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right) dy\, dz + dF_B = -\left(\tau_{yx} - \frac{d\tau_{yx}}{dy}\frac{dy}{2}\right) dx\, dz$$

$$\boxed{\frac{d\tau_{yx}}{dy} = \frac{\partial p}{\partial x} = \text{constant}}$$

Taylor Series Expansion



(*a*) Geometry of CV (*b*) Forces acting on CV

Fig. 8.3 Control volume for analysis of laminar flow between stationary infinite parallel plates.

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Fully Developed Internal Flows

Parallel Stationary Plates with Pressure Gradient (Poiseuille Flow)

- Both plates are stationary.
- The fluid is driven by pressure gradient.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions at $y=0 \quad u=0$

  at $y=a \quad u=0$

$$\frac{d\tau_{yx}}{dy}=\frac{\partial p}{\partial x}=\text{constant}$$

$$\tau_{yx}=\mu\frac{du}{dy}$$

Integrating twice and apply the boundary conditions

$$\boxed{u=\frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2-\left(\frac{y}{a}\right)\right]}$$

Velocity Profile

Taylor Series Expansion

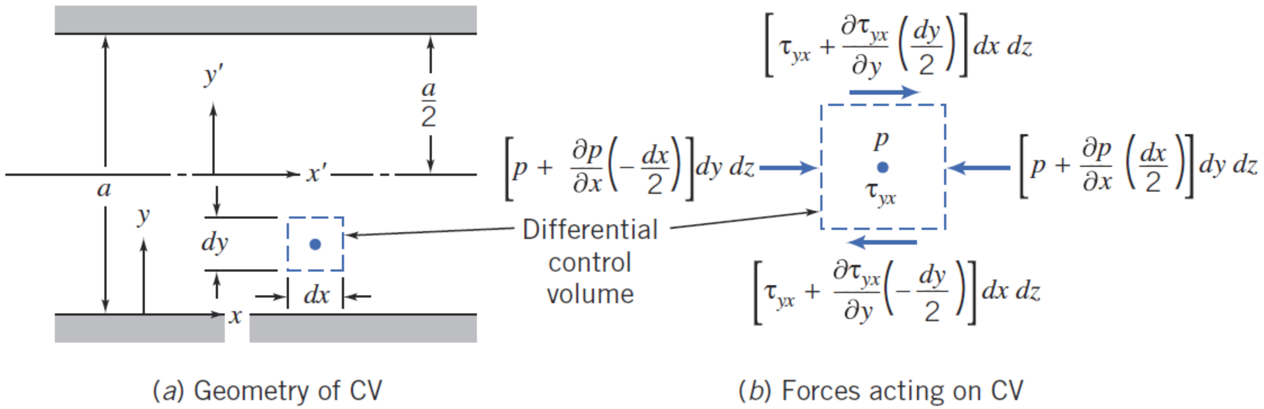

Fig. 8.3 Control volume for analysis of laminar flow between stationary infinite parallel plates.

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Fully Developed Internal Flows

Parallel Stationary Plates with Pressure Gradient (Poiseuille Flow)

- Both plates are stationary.
- The fluid is driven by pressure gradient.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions at $y=0 \quad u=0$
  at $y=a \quad u=0$

$$\boxed{\tau_{yx}=\mu\frac{\partial u}{\partial y}=a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a}-\frac{1}{2}\right]}$$

Shear Stress Distribution

$$Q=\int_0^a u l dy \text{ or } \frac{Q}{l}=\int_0^a \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2-ay)dy$$

Taylor Series Expansion

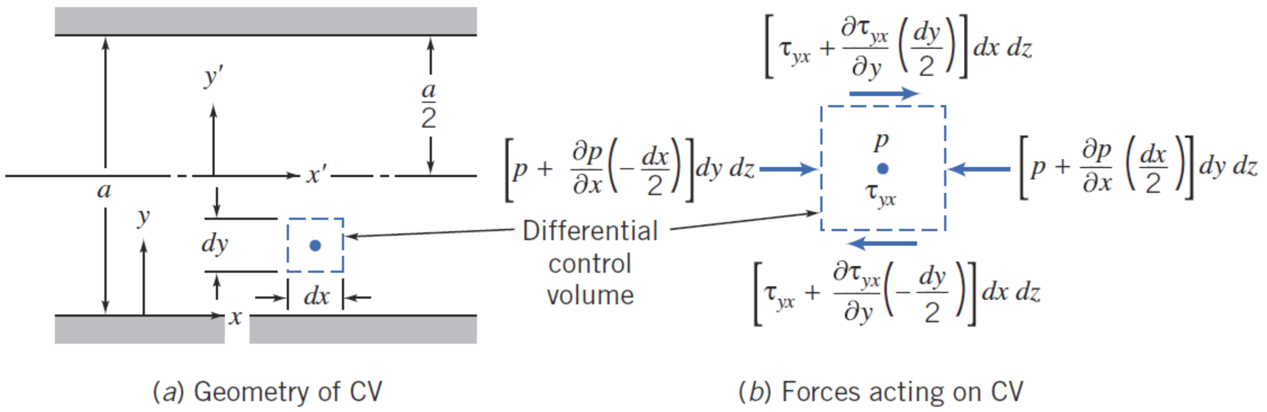

Fig. 8.3 Control volume for analysis of laminar flow between stationary infinite parallel plates.

$$\boxed{\frac{Q}{l}=-\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3=\frac{a^3\Delta p}{12\mu L}\text{ (constant pressure gradient)}}$$

Volume Flow Rate

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Fully Developed Internal Flows

Parallel Stationary Plates with Pressure Gradient (Poiseuille Flow)

- Both plates are stationary.
- The fluid is driven by pressure gradient.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions at $y=0 \quad u=0$

  at $y=a \quad u=0$

$$\boxed{\bar{V}=\frac{Q}{A}=\frac{a^2\Delta p}{12\mu L}\ (\text{constant pressure gradient})}$$

Average Velocity

$$\boxed{u=\frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2-\left(\frac{y}{a}\right)\right]\overset{y=a/2}{=\!=\!=}u_{\max}=-\frac{1}{8\mu}\left(\frac{\partial p}{\partial x}\right)a^2=\frac{3}{2}\bar{V}}$$

Maximum Velocity

NOTE: ALL this results are valid for laminar flow only!

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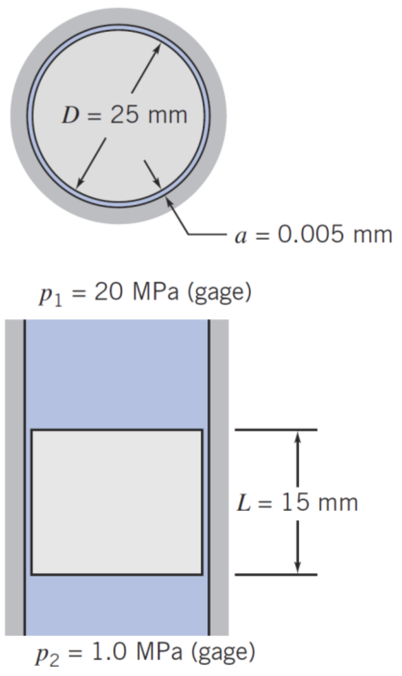
Fully Developed Internal Flows

Example 8.1

- A hydraulic system operates at a gage pressure of 20 MPa and 55℃ with a hydraulic fluid of SAE 10W oil. A control valve consists of a piston 25 mm in diameter, fitted to a cylinder with a mean radial clearance of 0.005 mm. Determine the leakage flow rate if the gage pressure on the low pressure side of the piston is 1.0 MPa and the piston is 15 mm long.



$$\frac{Q}{l} = \frac{a^3 \Delta p}{12 \mu L}$$

$$Q = \frac{a^3 l \Delta p}{12 \mu L} = \frac{a^3 \pi D \Delta p}{12 \mu L}$$

$$Q = \frac{\pi}{12} \times 0.005^3 \times 25 \times (20-1) \times 10^6 \times \frac{1}{0.018} \times \frac{1}{15 \times 10^{-3}} = 57.6 \ \mathrm{mm}^3/\mathrm{s}$$

NOTE: Remember to double check the Reynolds number.

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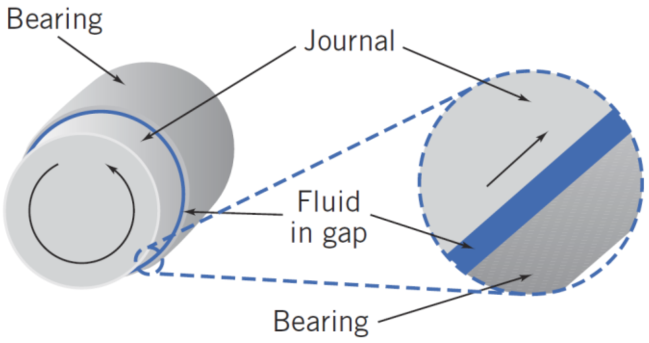
  - **Upper Moving Plates with Pressure Gradient**
  - **Pipe Flow**

Fully Developed Internal Flows

Upper Moving Plates with Pressure Gradient

- The fluid is driven by pressure gradient and the moving plate.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions $u=0$ at $y=0$
  $u=U$ at $y=a$

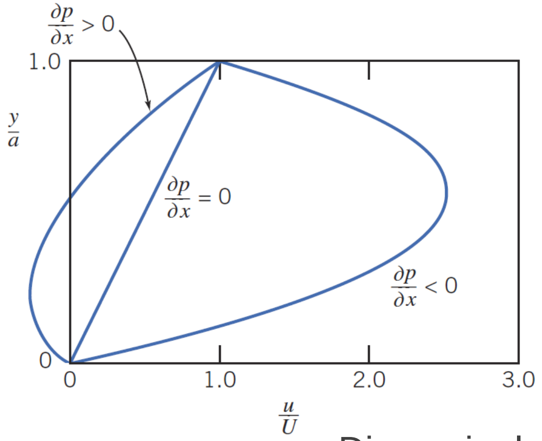


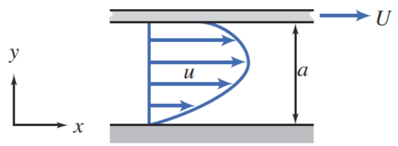
$$u=\frac{Uy}{a}+\frac{a^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left[\left(\frac{y}{a}\right)^2-\left(\frac{y}{a}\right)\right]$$ Velocity Profile

$$\tau_{yx}=\mu\frac{\partial u}{\partial y}=\mu\frac{U}{a}+a\left(\frac{\partial p}{\partial x}\right)\left[\frac{y}{a}-\frac{1}{2}\right]$$ Shear Stress Distribution

$$\frac{Q}{l}=\frac{Ua}{2}-\frac{1}{12\mu}\left(\frac{\partial p}{\partial x}\right)a^3$$ Volume Flow Rate

$$\frac{du}{dy}=0 \quad \text{at} \quad y=\frac{a}{2}-\frac{U/a}{(1/\mu)(\partial p/\partial x)}$$ Point of Maximum Velocity





Dimensionless Velocity Profile

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Fully Developed Internal Flows

Pipe Flow

- The fluid is driven by pressure gradient in a pipe.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions $u=0$ at $r=R$ (no slip condition)

$$dF_L = p2\pi r\,dr + dF_O = \left(\tau_{rx} + \frac{d\tau_{rx}}{dr}dr\right)2\pi(r+dr)dx$$

$$=$$

$$dF_R = -\left(p + \frac{\partial p}{\partial x}dx\right)2\pi r\,dr + dF_I = -\tau_{rx}2\pi r\,dx$$

$$\boxed{\frac{\partial p}{\partial x} = \frac{\tau_{rx}}{r} + \frac{d\tau_{rx}}{dr} = \frac{1}{r}\frac{d(r\tau_{rx})}{dr}}$$

$$u = \frac{r^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) + \frac{c_1}{\mu}\ln r + c_2 \longrightarrow \boxed{u = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]}$$

Velocity Profile

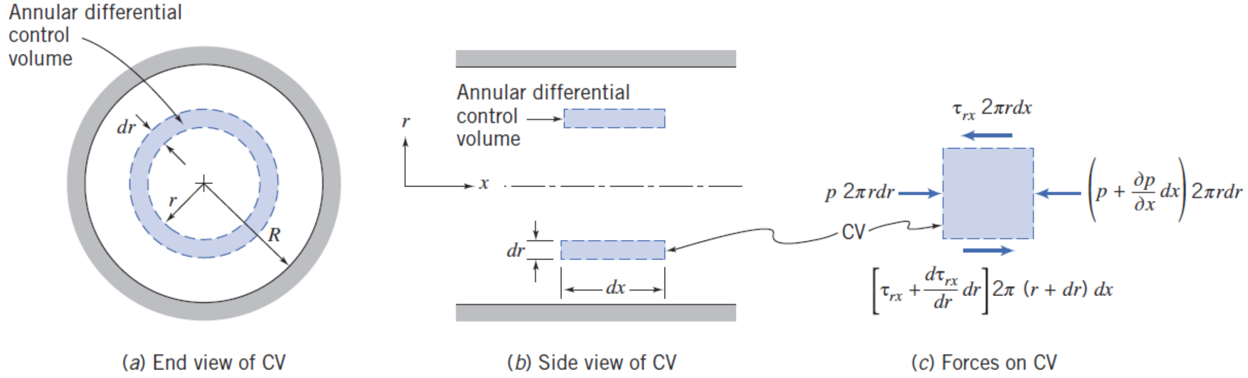

Fig. 8.7 Differential control volume for analysis of fully developed laminar flow in a pipe.

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Fully Developed Internal Flows

Pipe Flow

- The fluid is driven by pressure gradient in a pipe.
- Flow is laminar and fully developed, body force is neglected.
- Boundary conditions $u = 0$ at $r = R$ (no slip condition)

$$\boxed{\tau_{xy} = \mu \frac{du}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial x}\right)}$$

Shear Stress Distribution

$$\boxed{Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{\pi \Delta p R^4}{8\mu L} = \frac{\pi \Delta p D^4}{128\mu L}}$$

Volume Flow Rate

$$\boxed{\bar{V} = \frac{Q}{A} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)}$$

Averaged Velocity

$$\boxed{u_{\max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right) = 2\bar{V}}$$

Maximum Velocity

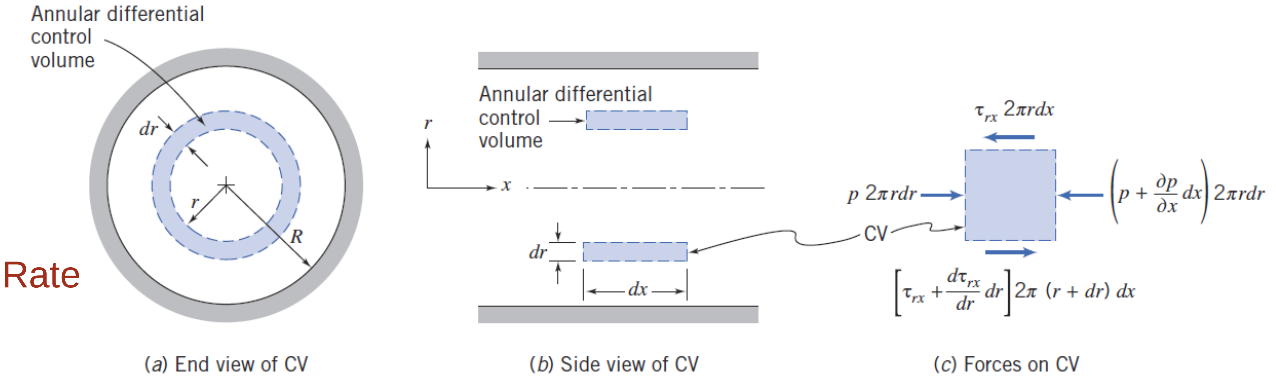

Fig. 8.7 Differential control volume for analysis of fully developed laminar flow in a pipe.

Friction lose and other loses caused by valves and elbows are neglected.

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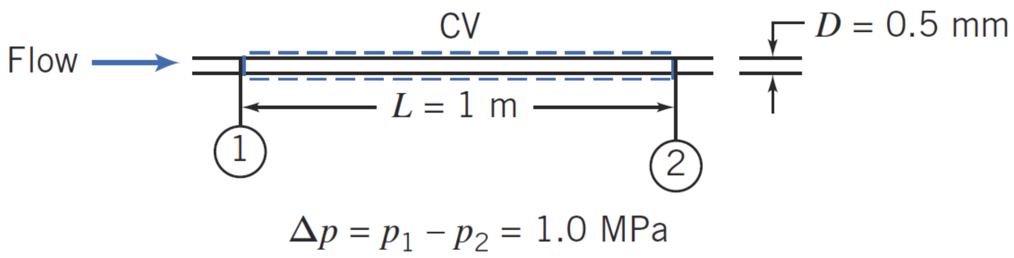
Fully Developed Internal Flows

Example 8.4

A test of a certain liquid in a capillary viscometer gave the following data:

$$Q = 880\frac{mm^3}{s}, \qquad L = 1\,m\,, \qquad D = 0.50\,mm, \qquad \Delta p = 1.0\,MPa$$

Determine the viscosity of the fluid.



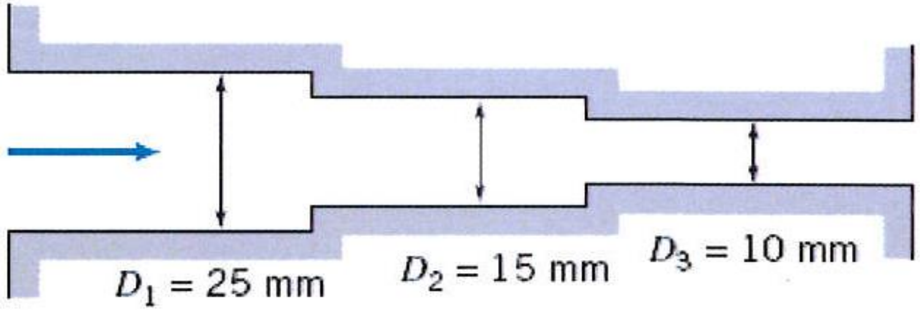
$$Q = \frac{\pi \Delta p D^4}{128\mu L}$$

$$\mu = \frac{\pi \Delta p D^4}{128 L Q} = \frac{\pi \times 1 \times 10^6 \times (0.5 \times 10^{-3})^4}{128 \times 1 \times 880 \times 10^{-9}} = 1.74 \times 10^{-3}\,\mathrm{N \cdot s/m^2}$$

Homework

Problem 8.3

8.3 Air at 40°C flows in a pipe system in which diameter is decreased in two stages from 25 mm to 15 mm to 10 mm. Each section is 2 m long. As the flow rate is increased, which section will become turbulent first? Determine the flow rates at which one, two, and then all three sections first become turbulent. At each of these flow rates, determine which sections, if any, attain fully developed flow.



$$Q_1 = \frac{Re_{crit}\pi v D_1}{4}$$

$$Q_2 = \frac{Re_{crit}\pi v D_2}{4}$$

$$Q_3 = \frac{Re_{crit}\pi v D_3}{4}$$

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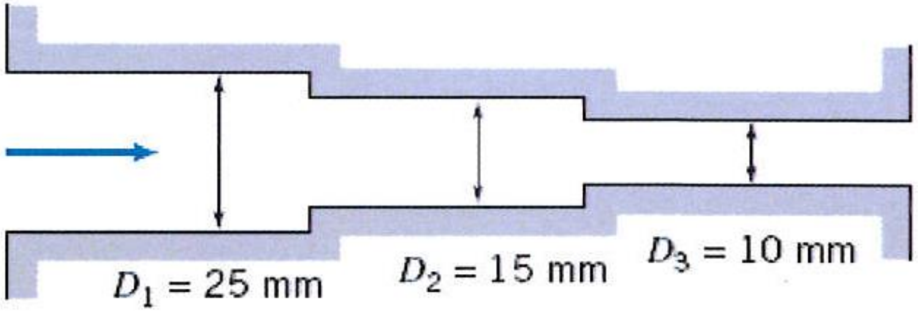
Homework

Problem 8.3



$$L_{laminar} = 0.06ReD, \quad Turbulent: L_{min} = 25D \quad L_{max} = 40D$$

$\boldsymbol{Q_3}$:

$$L_{min,3} = 25D_3 \qquad L_{max,3} = 40D_3$$

$$L_{laminar,3} = 0.06\left(\frac{4Q_3}{\pi v D_3}\right)D_3,$$

$$L_{laminar,2} = 0.06\left(\frac{4Q_3}{\pi v D_2}\right)D_2$$

$$L_{laminar,1} = 0.06\left(\frac{4Q_3}{\pi v D_1}\right)D_1$$

$\boldsymbol{Q_2}$:

$$L_{min,3} = 25D_3 \qquad L_{max,3} = 40D_3$$

$$L_{min,2} = 25D_2 \qquad L_{max,2} = 40D_2$$

$$L_{laminar,2} = 0.06\left(\frac{4Q_2}{\pi v D_2}\right)D_2$$

$$L_{laminar,1} = 0.06\left(\frac{4Q_2}{\pi v D_1}\right)D_1$$

$\boldsymbol{Q_1}$:

$$L_{min,3} = 25D_3 \qquad L_{max,3} = 40D_3$$

$$L_{min,2} = 25D_2 \qquad L_{max,2} = 40D_2$$

$$L_{min,1} = 25D_1 \qquad L_{max,1} = 40D_1$$

$$L_{laminar,1} = 0.06\left(\frac{4Q_1}{\pi v D_1}\right)D_1$$

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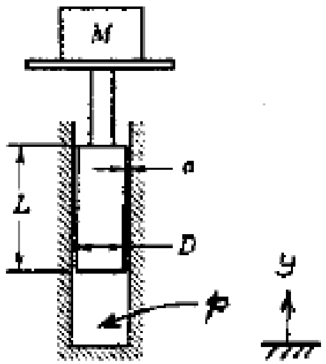
Homework

Problem 8.12

A large mass is supported by a piston of diameter *D* = 100 mm and length *L* = 100 mm. The piston sits in a cylinder closed at the bottom, and the gap *a* = 0.025 mm between the cylinder wall and piston is filled with SAE 10 oil at 20° C. The piston slowly sinks due to the mass, and oil is forced out at a rate of $6 \times 10^{-6}\ m^3/s$. What is the mass (kg)?



Flow between stationary plates:

$$V_{ave} = \frac{Q}{a\pi D} \quad \rightarrow \quad Re = \frac{aV_{ave}}{v}$$

$$\frac{Q}{l} = \frac{a^3 \Delta p}{12\mu L}$$

$$\Delta p \rightarrow F \rightarrow m$$

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Homework

Problem 8.18

Consider fully developed laminar flow between infinite parallel plates separated by gap width d = 10 mm. The upper plate moves to the right with speed U_2= 0.5 m/s; the lower plate moves to the left with speed U_1= 0.25 m/s. The pressure gradient in the direction of flow is zero. Develop an expression for the velocity distribution in the gap. Find the volume flow rate per unit depth (m^3/sec/m) passing a given cross section.

$$\frac{dP}{dx} = 0 \quad \rightarrow \quad \frac{d\tau}{dy} = \mu \frac{d^2u}{dy^2} = 0$$

$$\frac{d^2u}{dy^2} = 0 \quad \rightarrow \quad u(y), \qquad u(0) = -U_1, \qquad u(d) = U_2$$

$$Q = \int u\, dA = b\int_0^d u\, dy \quad \rightarrow \quad \frac{Q}{b} = ???$$

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Homework

Problem 8.39

Consider first water and then SAE 10W lubricating oil flowing at 40℃ in a 6 mm diameter tube. Determine the maximum flow rate and corresponding pressure gradient for each fluid assuming laminar flow.

$$Re = \frac{\rho \bar{V} D}{\mu} = \frac{4\mathrm{Q}}{\pi D v}$$

$$Q = Q_{max} = \frac{Re_{max} \pi D v}{4} = \left(\frac{Re_{max} \pi D}{4}\right) v$$

$$\frac{Q_{max}}{A} = -\frac{R^2}{8\mu}\frac{\delta p}{\delta x}$$

$$\frac{\delta p}{\delta x} = -\frac{128 \mu Q_{max}}{\pi D^4} = \left(-\frac{128}{\pi D^4}\right) Q_{max}\, \mu$$

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Homework

Problem 8.40

For fully developed laminar flow in a pipe, determine the radial distance from the pipe axis at which the velocity equals the average velocity.

$$\bar{V} = \frac{Q}{A} = -\frac{R^2}{8\mu}\frac{\delta p}{\delta x}$$

$$u = -\frac{R^2}{4\mu}\frac{\delta p}{\delta x}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

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Homework

This assignment is due by 6pm on March 18th.

- Upload your solution to BB.
- You can either type your solution out using a word editor like Microsoft Word or clearly hand write the information and then copy/scan your solution to a digital file.

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