Sichuan University - Pittsburgh Institute

ME 1071: Applied Fluids

Lecture 10 Introduction to Compressible Flow

Spring 2021

Weekly Study Plan

Weeks	Dates	Lectures
1	Mar. 9	Course Introduction, Fluids Review
2	Mar. 16	Chapter 8: Internal Incompressible Viscous Flow
3	Mar. 23	Chapter 8: Internal Incompressible Viscous Flow
4	Mar. 30	Chapter 8/Exam I Review
5	**Apr. 6**	**Exam I**
6	Apr. 13	Chapter 9: External Incompressible Viscous Flow
7	Apr. 20	Chapter 9: External Incompressible Viscous Flow
8	Apr. 25	Chapter 11: Flow in Open Channels
9	Apr. 27	Chapter 11: Flow in Open Channels
10	May. 11	Exam II Review
11	**May. 18**	**Exam II**
12	May. 25	Chapter 12: Introduction to Compressible Flow
13	Jun. 1	Chapter 12: Introduction to Compressible Flow
14	Jun. 8	Chapter 12: Introduction to Compressible Flow
15	Jun. 15	Chapter 5: CFD Related Topics
16	Jun. 22	Final Exam Review **(Final Exam at Jun. 26)**

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Outlines

- **When is a flow Compressible?**
- **Critical Conditions**
- **Basic Equations for One-Dimensional Compressible Flow**
- **Isentropic Flow**
  - **Area Change**
  - **Subsonic/Supersonic Flow**
- **Choked Flow**

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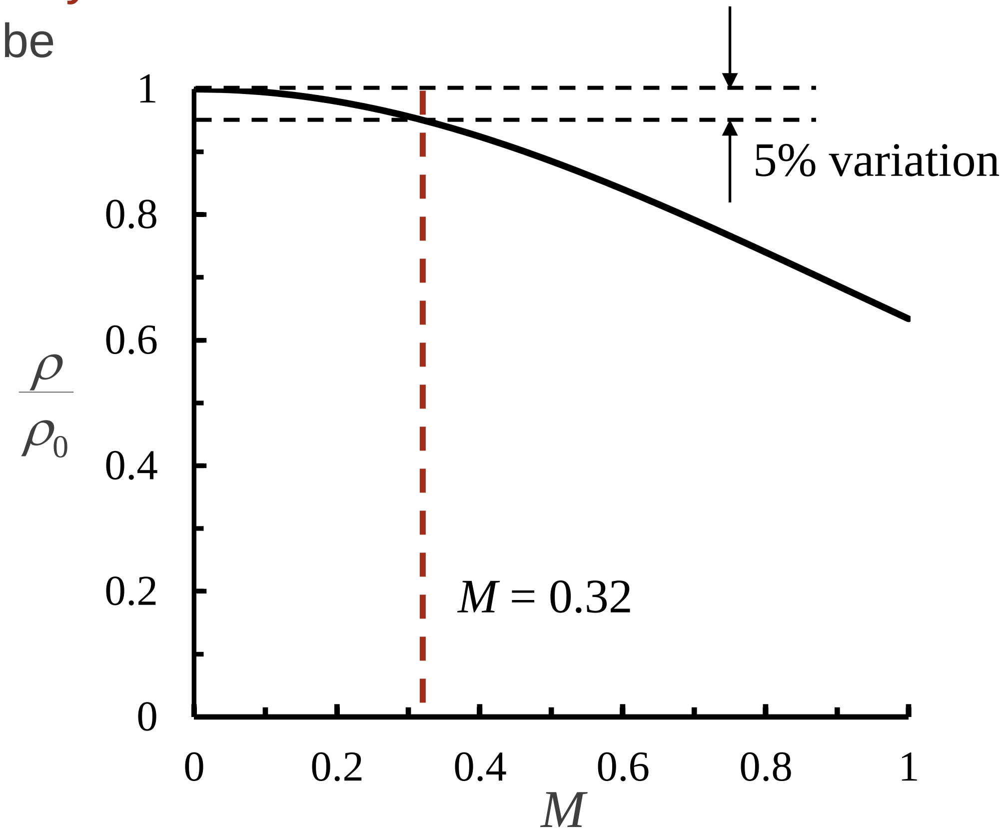
When Is a Flow Compressible?

- Compressible flow: fluid density changes significantly.
  - The relation between density and Mach number can be described by:

$$\frac{\rho_0}{\rho} = \left(1 + \frac{k-1}{2} M^2\right)^{1/(k-1)}$$

- To ensure density change < 5%, M must be less than 0.3.
- When M < 0.3, the flow can be treated as incompressible; otherwise, the compressibility must be considered.



Isentropic variation of density with Mach number

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When Is a Flow Compressible?

- **Example**
  - Consider the flow of air through a nozzle starting in the reservoir at nearly zero velocity and standard sea level values of p_0 = 1 atm and T_0 = 288 K, and expanding to a velocity of 107 m/s at the nozzle exit. Calculate the pressure at the nozzle exit assuming first incompressible flow and then compressible flow.

How about expanding to a velocity of 275 m/s?

incompressible

From Bernoulli's equation

$$p = p_0 - \frac{1}{2}\rho_0 V^2 = p_0 - \frac{1}{2}\frac{p_0}{RT_0}V^2 = 101325 - 0.5\frac{101325}{287\times 288}107^2 = 94307 \text{ Pa}$$

54972 Pa

compressible

$$T = T_0 - \frac{V^2}{2c_p} = 288 - \frac{107^2}{2(1.4\times 287/(1.4-1))} = 282.3 \text{ K}$$

250.4 K

$$p = p_0\left(\frac{T}{T_0}\right)^{k/(k-1)} = 101325\left(\frac{282.3}{288}\right)^{3.5} = 94478 \text{ Pa}$$

88096 Pa

$$M = V/c = V/\sqrt{kRT} = 107/\sqrt{1.4\times 287\times 282.3} = 0.317$$

0.866

The flow can be treated as incompressible.

The flow is compressible.

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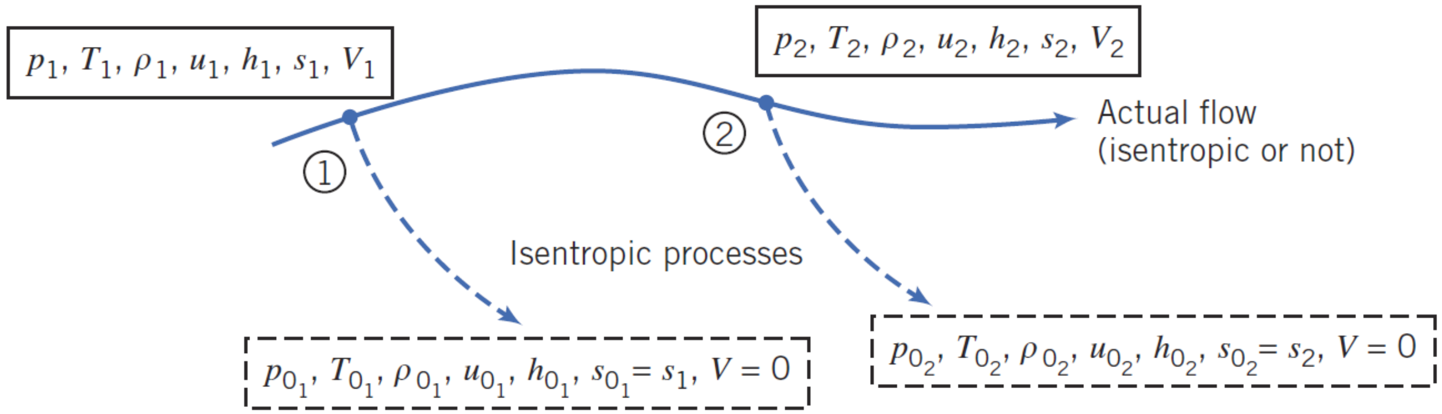
Critical Conditions

- **Sonic condition (音速状态) / Critical Condition**
  - The condition that the velocity of fluid element adiabatically or isentropically approaches to sonic velocity (M = 1).

$$\overset{V}{\boxed{h,\ T}} \xrightarrow{\text{Approaches adiabatically}} \overset{V^* = c^*}{\boxed{h^*,\ T^*}} \qquad \overset{V}{\boxed{p,\ \rho}} \xrightarrow{\text{Approaches isentropically}} \overset{V^* = c^*}{\boxed{p^*,\ \rho^*}}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1} \qquad c^* = \sqrt{kRT^*} = \sqrt{\frac{2k}{k+1}RT_0}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \qquad \frac{\rho^*}{\rho_0} = \left(\frac{2}{k+1}\right)^{1/(k-1)}$$



Local isentropic stagnation properties.

The critical conditions are similar to the stagnation conditions, except that the final velocity is brought to sonic velocity (M = 1) instead of zero velocity.

Outlines

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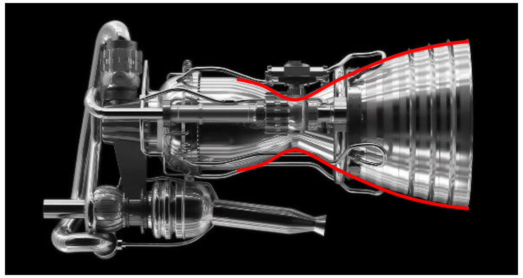
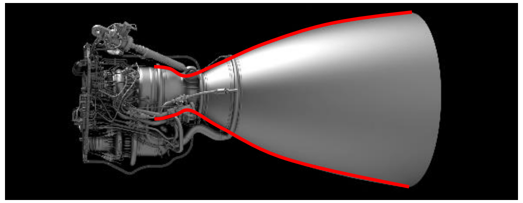
Introduction

- **Compressible Flow Through Ducts (槽道)**
  - Nozzle (喷管): A duct to increase the flow velocity in the expense of pressure or internal energy.
  - Diffuser (扩压器): A duct to decrease the flow velocity.
  - Wind tunnel (风洞): combination of nozzles and diffusers to provide uniform supersonic flow for testing.
  - Quasi-one-dimensional flow (准一维流动): one with varied cross-sectional area in which all variables vary primarily along one direction.

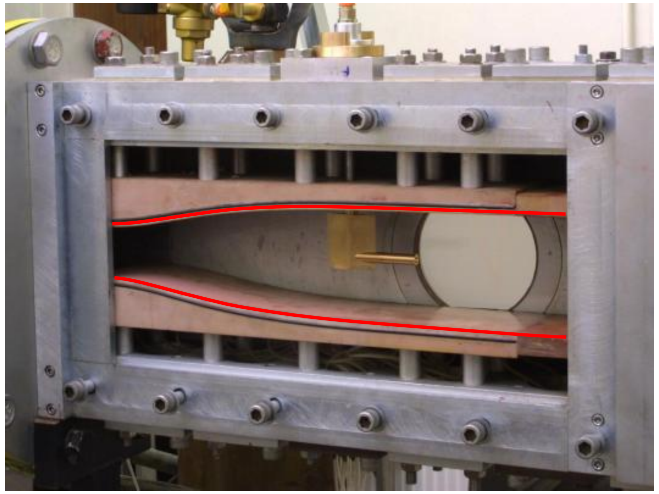
SpaceX's Merlin Engines in Falcon Heavy



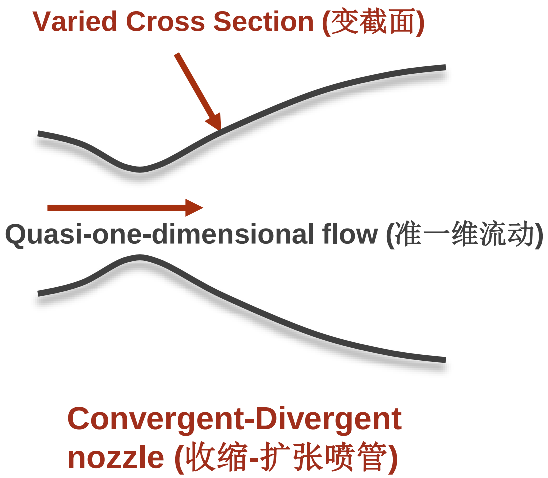
SpaceX's Merlin Engine



SpaceX's Raptor Engine



Supersonic Wind Tunnel of Imperial College London



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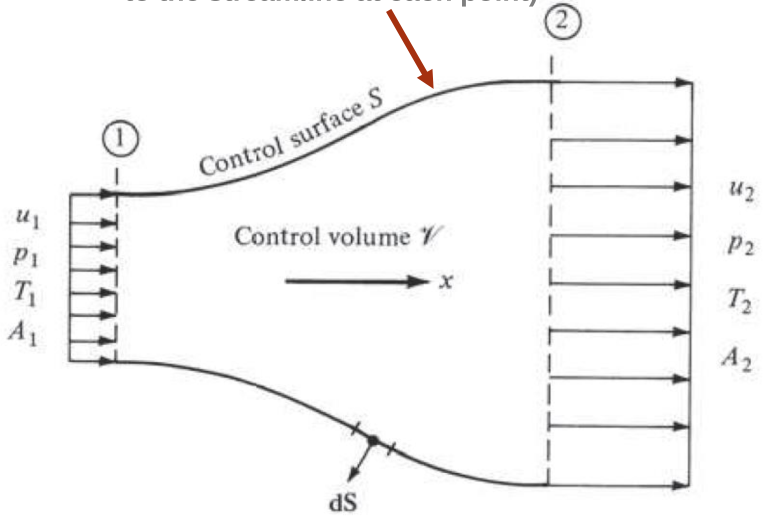
Basic Equations for One-Dimensional Compressible Flow

- **Quasi-one-dimensional flow**
  - Flow properties are uniform across any cross section at a given x station.
  - The slope/gradient of the changes in area is small and smooth.
- **Continuity Equation**

Streamline (velocity is tangential to the streamline at each point)



- **Assumptions**
  - Steady flow, $\partial / \partial t = 0$.
  - Adiabatic.
  - No viscous effects.
  - No body forces, $\mathbf{f} = 0$.
  - Quasi-one-dimensional flow
  - ~~$M_1 >= 1$~~ (not required)

Integration over the surface of the control volume

$$\boxed{\oiint_S \rho \mathbf{V} \bullet \mathbf{dS} = 0}$$

$$-\rho_1 u_1 A_1 + 0 + \rho_2 u_2 A_2 + 0 = 0$$

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Continuity equation for Quasi-one-dimensional flow

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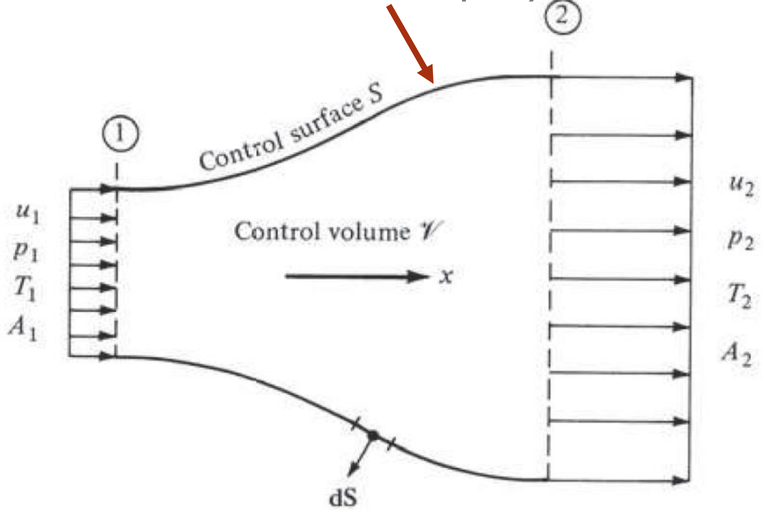
Basic Equations for One-Dimensional Compressible Flow

- **Momentum Equation**
  - Only one direction, x.
  - dA is the x component of the vector $\mathbf{dS}$.
  - Additional pressure force term due to area variation.

Streamline (velocity is tangential to the streamline at each point)



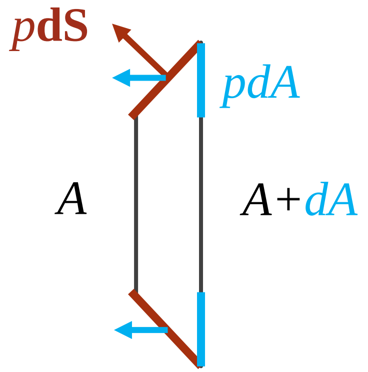
$$\oiint_S (\rho \mathbf{V}\cdot \mathbf{dS})\mathbf{V} = -\oiint_S p\,\mathbf{dS}$$

$$\oiint_S (\rho \mathbf{V}\cdot \mathbf{dS})u = -\oiint_S (p\,dS)_x$$

$$-\rho_1 u_1 A_1 u_1 + 0 + \rho_2 u_2 A_2 u_2 + 0 = -\left(-p_1 A_1 + p_2 A_2 - \int_{A_1}^{A_2} p\,dA\right)$$

$$\boxed{p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = p_2 A_2 + \rho_2 u_2^2 A_2}$$

Momentum equation for Quasi-one-dimensional flow



For upper and lower surface

$$-\oiint_S (p\,dS)_x = -\int_{A_1}^{A_2} p\,dA$$

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Basic Equations for One-Dimensional Compressible Flow

- **Energy Equation**
  - The general result for steady, inviscid, adiabatic flow with the total enthalpy being constant.

Energy conservation

$$\oiint_S (\rho \mathbf{V} \cdot \mathbf{dS})\left(e+\frac{V^2}{2}\right) = -\oiint_S (p\mathbf{dS})\cdot\mathbf{V}$$

$$-\rho_1 u_1 A_1\left(e_1+\frac{u_1^2}{2}\right)+\rho_2 u_2 A_2\left(e_2+\frac{u_2^2}{2}\right) = -(-\rho_1 u_1 A_1 + \rho_2 u_2 A_2)$$

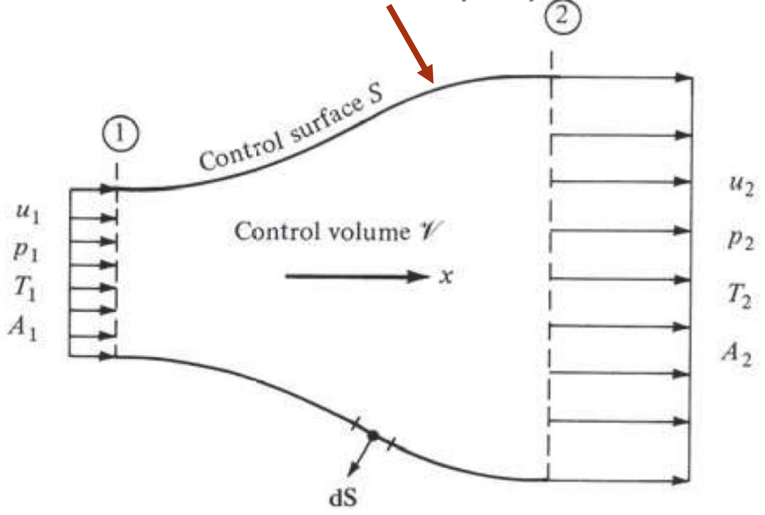
Continuity

$$\boxed{h_1+\frac{u_1^2}{2}=h_2+\frac{u_2^2}{2}}$$

Energy equation for Quasi-one-dimensional flow

$$\boxed{h_{01}=h_{02} \rightarrow T_{01}=T_{02}}$$

Streamline (velocity is tangential to the streamline at each point)



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Basic Equations for One-Dimensional Compressible Flow

- **Continuity**

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

- **Momentum**

$$\boxed{p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\,dA = p_2 A_2 + \rho_2 u_2^2 A_2}$$

- **Energy**

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}}$$

- **Enthalpy**

$$\boxed{h_2 = c_p T_2}$$

- **Equation of state**

$$\boxed{p_2 = \rho_2 R T_2}$$

- **Assumptions**
  - Steady flow, $\partial / \partial t = 0$.
  - Adiabatic.
  - No viscous effects.
  - No body forces, $\mathbf{f} = 0$.
  - Quasi-one-dimensional flow
  - ~~$M_1 >= 1$~~ (not required)
  - Area changes are prescribed as $A(x)$.

Five equations with five unknowns.

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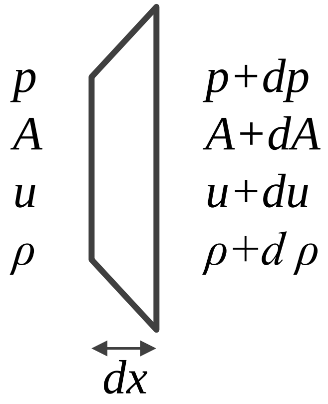
Basic Equations for One-Dimensional Compressible Flow

- **Governing Equations in Differential Form (微分形式的控制方程)**
  - Convert the five governing equations in integral form to differential form



Momentum

$$\boxed{p_1A_1+\rho_1u_1^2A_1+\int_{A_1}^{A_2} p\,dA=p_2A_2+\rho_2u_2^2A_2}$$

$$pA+\rho u^2A+pdA=(p+dp)(A+dA)+(\rho+d\rho)(u+du)^2(A+dA)$$

$$Adp+Au^2d\rho+\rho u^2dA+2\rho uAdu=0$$

$$d(\rho uA)=0$$

$$u^2Ad\rho+\rho uAdu+\rho u^2dA=0$$

$$\boxed{dp=-\rho udu}$$

Euler's equation

Continuity

$$\boxed{\rho_1u_1A_1=\rho_2u_2A_2}$$

$$\rho uA=\text{const}$$

$$\boxed{d(\rho uA)=0}$$

$$d(\rho u)=0$$

One-dimensional Flow

Energy

$$\boxed{h_1+\frac{u_1^2}{2}=h_2+\frac{u_2^2}{2}}$$

$$h+\frac{u^2}{2}=\text{const}$$

$$\boxed{dh+udu=0}$$

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Isentropic Flow

- **Area-velocity relation**
  - Further simplify the differential equations to obtain some physical insights into the quasi-one-dimensional flow.

Continuity

$$\boxed{d(\rho uA)=0} \xrightarrow{uAd\rho+\rho Adu+\rho udA=0} \frac{d\rho}{\rho}+\frac{du}{u}+\frac{dA}{A}=0$$

Euler's equation

$$\boxed{dp=-\rho udu}$$

$$\frac{dp}{\rho}=\frac{dp}{d\rho}\frac{d\rho}{\rho}=-udu$$

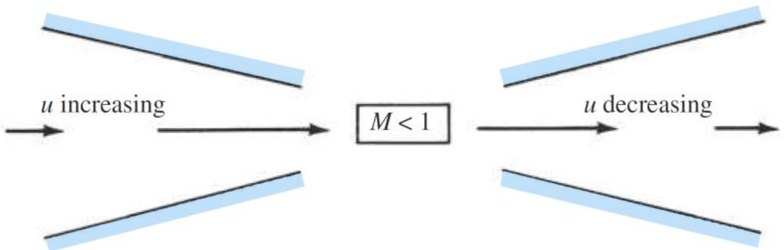
Isentropic assumption

$$\frac{dp}{d\rho}\equiv\left(\frac{\partial p}{\partial\rho}\right)_s=a^2$$

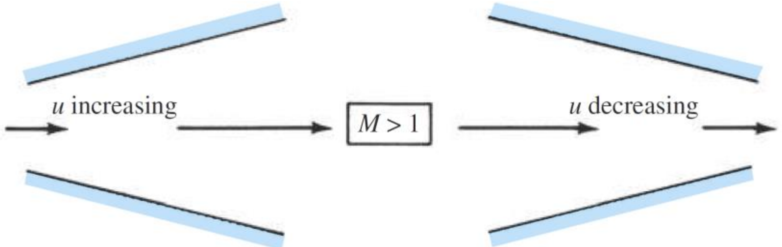
$$\frac{d\rho}{\rho}=-\frac{u^2}{a^2}\frac{du}{u}=-M^2\frac{du}{u}$$

$$\boxed{\frac{dA}{A}=(M^2-1)\frac{du}{u}}$$

Area-velocity relation



- $M < 1$, subsonic flow, $du \propto -dA$
- $M > 1$, supersonic flow, $du \propto dA$
- $M = 1$, $dA = 0$, minimum area
- $M = 0$, incompressible flow, Au = constant.



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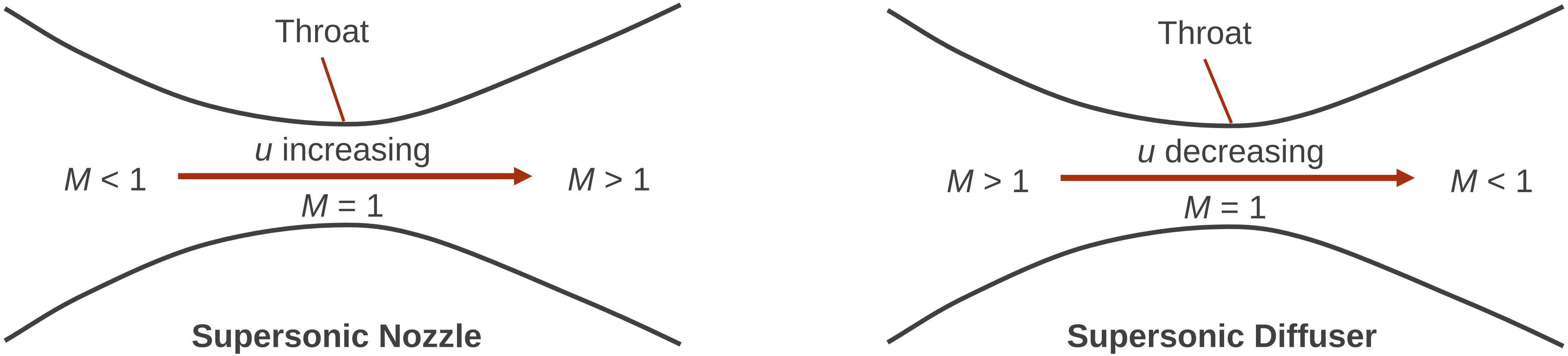
Governing Equations for Quasi-One-Dimensional Flow

- **Operation of Convergent-Divergent duct**
  - Throat (喉道): the minimum area of the convergent-divergent duct.
  - Supersonic Nozzle : convergent-divergent duct operated with a subsonic inflow ($M < 1$).
  - Supersonic Diffuser: convergent-divergent duct operated with a supersonic inflow ($M > 1$).
  - Supersonic nozzle or supersonic diffuser depends on inflow Mach number.



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Isentropic Flows

- **Area-Mach number relation**
  - The ratio of A* and A can be express as a function of Mach number in nozzle flows (inflow Mach number M < 1).

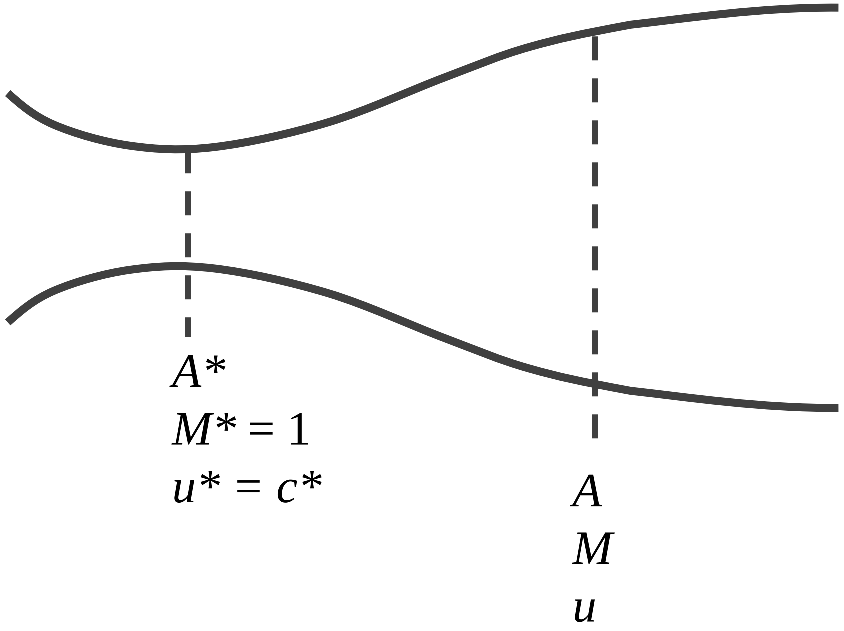
$$\rho^* u^* A^* = \rho u A$$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{u^*}{u} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{c^*}{u}$$

$$\left(\frac{A}{A^*}\right)^2 = \left(\frac{\rho^*}{\rho_0}\right)^2 \left(\frac{\rho_0}{\rho}\right)^2 \left(\frac{c^*}{u}\right)^2$$

$$\left\{ \begin{aligned} \frac{\rho^*}{\rho_0} &= \left(\frac{2}{k+1}\right)^{1/(k-1)} \\ \frac{\rho_0}{\rho} &= \left(1 + \frac{k-1}{2}M^2\right)^{1/(k-1)} \\ M^{*2} &= \frac{(k+1)M^2}{2+(k-1)M^2} \end{aligned} \right.$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{k+1}\left(1+\frac{k-1}{2}M^2\right)\right]^{(k+1)/(k-1)}$$



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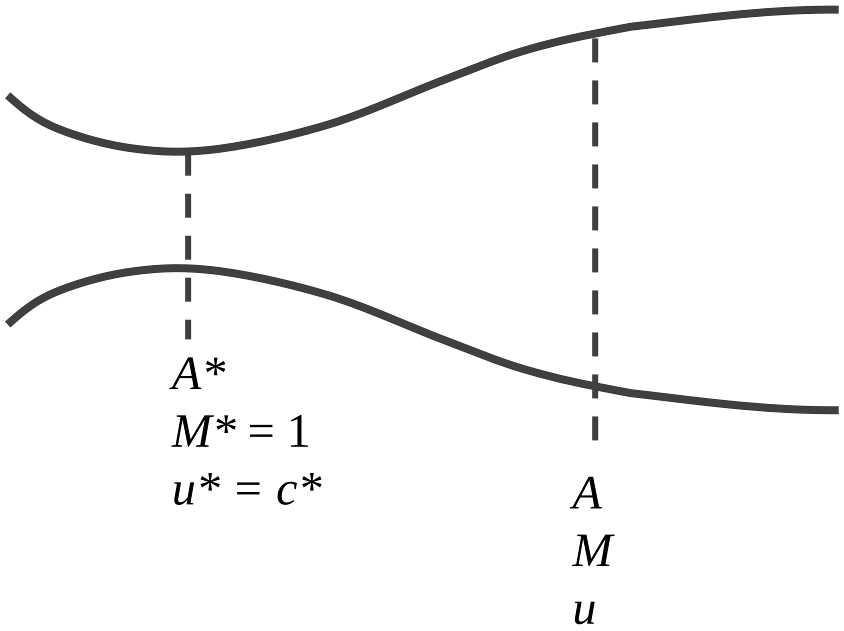
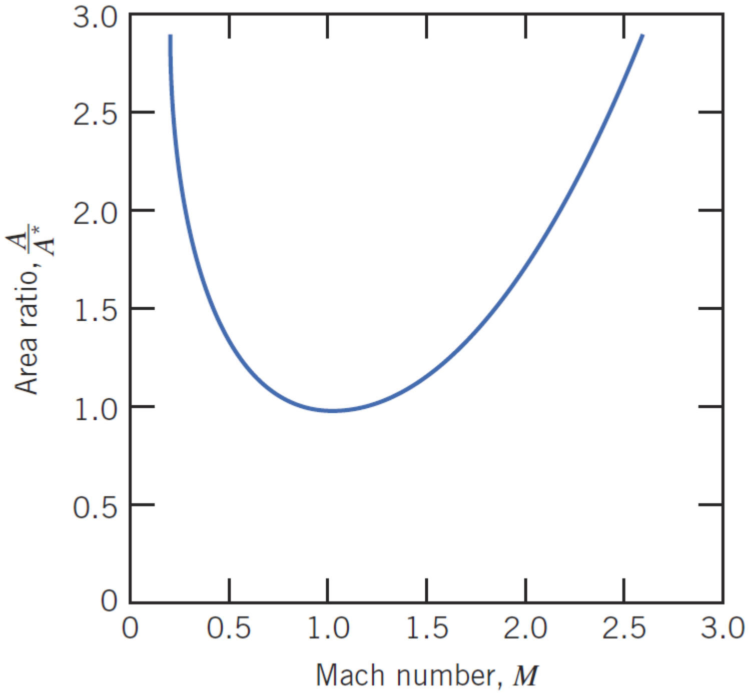
Isentropic Flows

- **Area-Mach number relation**
  - The ratio of A* and A can be express as a function of Mach number in nozzle flows (inflow Mach number M < 1).

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{k+1}\left(1+\frac{k-1}{2}M^2\right)\right]^{(k+1)/(k-1)}$$





The Mach number inside a supersonic nozzle is a function of area ratio only!

Isentropic Flows

Example

Air flows isentropically in a channel. At section (1) the Mach number is 0.3, the area is 0.001 m^2, and the absolute pressure and the temperature are 650 kPa and 62℃, respectively. At section (2), the Mach number is 0.8. Evaluate the properties at section (2).

$$T_{0_1} = T_1\left(1 + \frac{k-1}{2}M_1^2\right) = T_{0_2}$$

$$p_{0_1} = p_1\left[1 + \frac{k-1}{2}M_1^2\right]^{k(k-1)} = p_{0_2} \qquad p_2 = \frac{p_{02}}{\left[1 + \frac{k-1}{2}M_1^2\right]^{k(k-1)}}$$

$$T_2 = \frac{T_{02}}{\left(1 + \frac{k-1}{2}M_1^2\right)}$$

$$\rho_2 = \frac{p_2}{RT_2} \qquad V_2 = M_2 c_2 = M_2\sqrt{kRT_2}$$

$$\frac{A_2}{A_1} = \frac{A_2 A^*}{A^* A_1} = \left\{\frac{1}{M_2}\frac{\left[1 + \frac{k-1}{2}M_2^2\right]^{\frac{k+1}{2(k-1)}}}{\frac{k+1}{2}}\right\} \Bigg/ \left\{\frac{1}{M_1}\frac{\left[1 + \frac{k-1}{2}M_1^2\right]^{\frac{k+1}{2(k-1)}}}{\frac{k+1}{2}}\right\}$$

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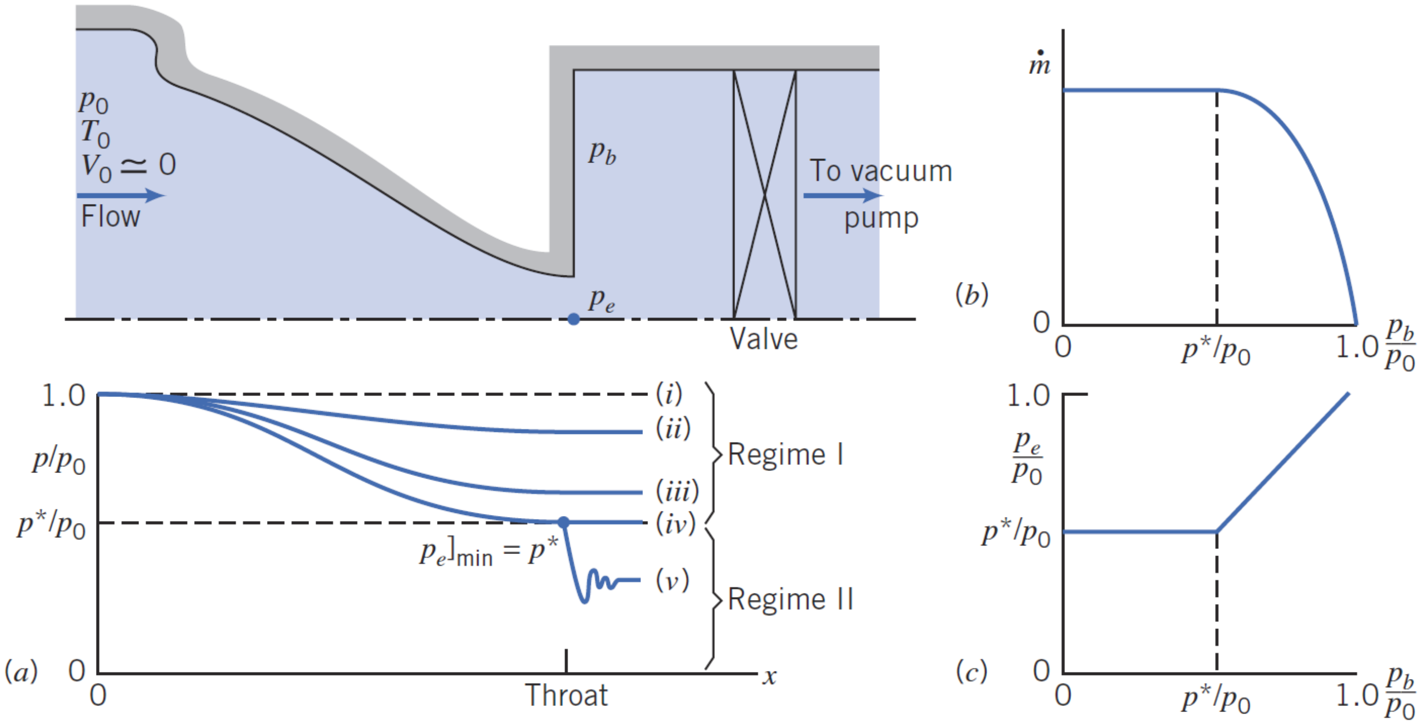
Choked Flow

- **The Flow Rate of an Isentropic Flow with Area Variation**
  - The limiting of the mass flow rate is called choking of the flow, this happens when the Mach number at the throat equals to 1.



$$\dot{m}_{\text{choked}} = \rho^* c^* A_e$$

$$c^* = \sqrt{kRT^*} = \sqrt{\frac{2k}{k+1} RT_0}$$

$$\dot{m}_{\text{choked}} = A_e p^* \sqrt{\frac{k}{RT^*}}$$

$$\left.\frac{p_e}{p_0}\right|_{\text{choked}} = \frac{p^*}{p_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)}$$

$$\frac{T^*}{T_0} = \frac{2}{k+1}$$

$$\boxed{\dot{m}_{\text{choked}} = A_e p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)}}$$

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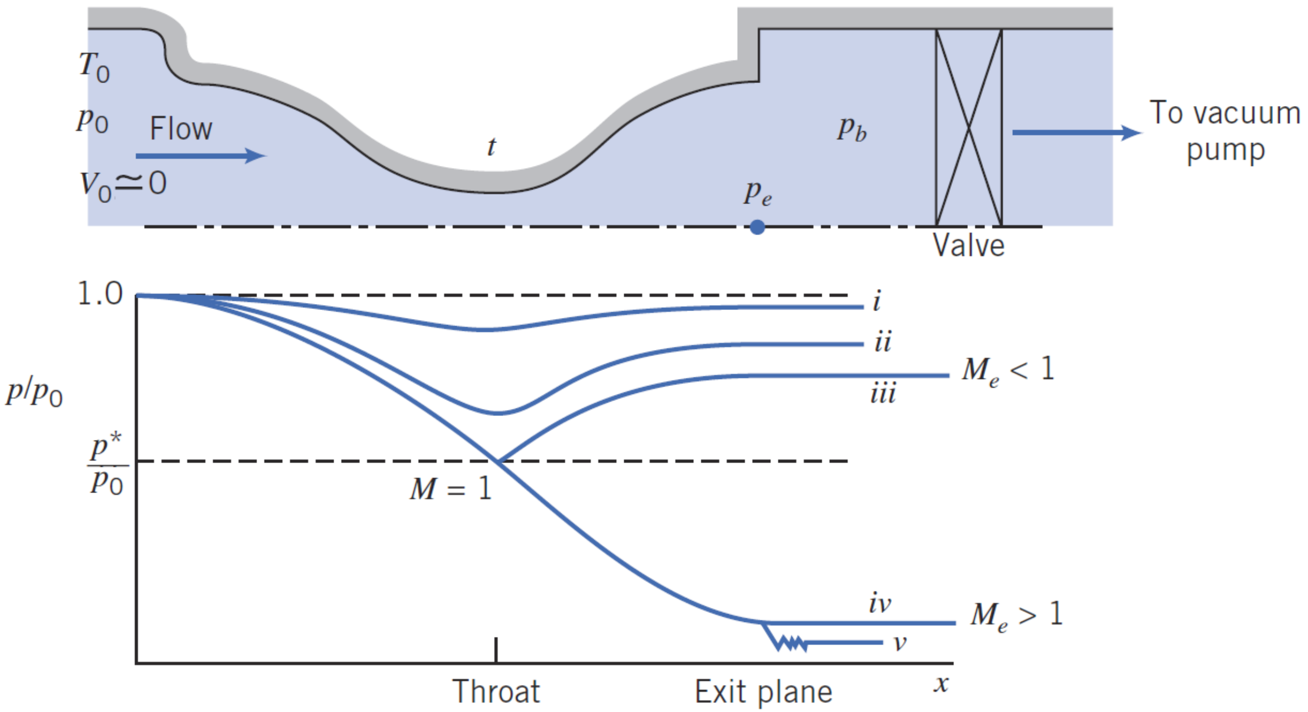
Choked Flow

- **Throat**: the minimum area of the convergent-divergent duct.



Pressure distributions for isentropic flow in a converging-diverging nozzle.

$$\boxed{\dot{m}_{\text{choked}} = A_t p_0 \sqrt{\frac{k}{RT_0}} \left(\frac{2}{k+1}\right)^{(k+1)/2(k-1)}}$$

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Homework

Problem 12.34

Consider steady, adiabatic flow of air through a long straight pipe with A = 0.05 m^2. At the inlet the air is at 200 kPa, 60℃ and 146 m/s. Downstream at section 2, the air is at 95.6 kPa and 280 m/s. Determine $p_{o_1}, p_{0_2}, T_{0_1}, T_{0_2}$ and the entropy change for the flow.

$$\frac{dQ}{dm} = h_{0_2} - h_{0_1} = 0$$

$$h_{0_2} - h_{0_1} = c_p\left(T_{0_2} - T_{0_1}\right) = 0$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} = h_{0_1} = h_{0_2}$$

$$\rho_1 V_1 A = \rho_2 V_2 A$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{P_2}{P_1}$$

$$s_{0_2} - s_{0_1} = -R\ln\frac{P_{0_2}}{P_{0_1}}$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)}$$

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Homework

Problem 12.38

Air flows from the atmosphere into an evacuated tank through a convergent nozzle of 38 mm tip diameter. If atmospheric pressure and temperature are 101.3 kPa and 15 ℃, respectively, what vacuum must be maintained in the tank to produce sonic velocity in the jet? What is the flow rate? What is the flow rate when the vacuum is 254 mm of mercury?

$$Vac = (P_0 - P) \times \frac{760\ mm}{101.3\ kPa}$$

$$c = \sqrt{kRT} = V$$

$$\dot{m} = \rho V A$$

$$h_0 = h + \frac{1}{2}V^2$$

$$\dot{m} = \rho V A$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)}$$

Homework

Problem 12.43

Nitrogen flows through a diverging section of duct with $A_1 = 0.15\ m^2$ and $A_2 = 0.45\ m^2$. If $M_1 = 0.7$ and $P_1 = 450\ kPa$, find M_2 and P_2.

$$p_{0_1} = p_{0_2}$$

$$A_1^* = A_2^*$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2} M^2\right]^{k/(k-1)}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2} M^2$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2} M^2\right]^{1/(k-1)}$$

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}}\right]^{(k+1)/2(k-1)}$$

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Homework

Problem 12.46

Air, at an absolute pressure of 60.0 kPa and 27℃ enters a passage at 486 m/s, where $A = 0.02\ m^2$. At section 2 downstream, $p = 78.8\ kPa$. Assuming isentropic flow, calculate the Mach number at section 2. Sketch the flow passage.

$$p_{0_1} = p_{0_2}$$

$$c = \sqrt{kRT}$$

$$M_1 = \frac{V_1}{c_1}$$

$$\frac{p_0}{p} = \left[1 + \frac{k-1}{2}M^2\right]^{k/(k-1)}$$

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2$$

$$\frac{\rho_0}{\rho} = \left[1 + \frac{k-1}{2}M^2\right]^{1/(k-1)}$$

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Homework

This assignment is due by **6pm on June 3rd**.

- Upload your solution to BB.
- You can either type your solution out using a word editor like Microsoft Word or clearly hand write the information and then copy/scan your solution to a digital file.

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