ME 1049

Mechatronics Lab

Inverted Pendulum Control

Revised May

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Mechanical Engineering Department

Lab 7: Inverted Pendulum Control

Figure 0: A landing rocket can be viewed as a similar control challenge to an inverted pendulum

Balancing an inverted pendulum may seem like a purely academic challenge, but the control algorithm that is used is analogous to a wide variety of problems from Segway scooters to rockets. More broadly, the stabilization of unstable systems is a ubiquitous problem that requires much the same approach to controller design and development as the approaches covered in this lab. Complex systems often require more complex controllers to achieve a desired level of performance, but ultimately the approaches presented in this lab represent an important collection of skills for any modern control systems engineer.

Learning Objectives

After completing this lab, you should be able to complete the following activities.

1. Introduction to state-space models
2. Design a controller to balance an inverted pendulum using pole placement
3. Design an optimized controller for an inverted pendulum
4. Implement a hybrid swing-up controller for energy-based automatic inversion

Required Tools and Technology

Platform: NI ELVIS III	✓ View the NI ELVIS III User Manual http://www.ni.com/en-us/support/model.ni-elvis-iii.html
Hardware: Quanser Controls Board	✓ View the Controls Board User Manual http://www.ni.com/en-us/support/model.quanser-controls-board-for-ni-elvis-iii.html
Software: LabVIEW Version 18.0 or Later Toolkits and Modules: • LabVIEW Real-Time Module • NI ELVIS III Toolkit • LabVIEW Control Design & Simulation	• Before downloading and installing software, refer to your professor or lab manager for information on your lab's software licenses and infrastructure • Download & Install for NI ELVIS III http://www.ni.com/academic/download • View Tutorials http://www.ni.com/academic/students/learn-labview/

Expected Deliverables

In this lab, you will collect the following deliverables:

- ✓ States-space model of the pendulum
- ✓ Calculated control gains to meet the specifications
- ✓ Simulated response to the designed control gains
- ✓ Measured response on hardware including performance analysis
- ✓ Optimal controller design for the inverted pendulum
- ✓ Understanding of the effect of the design parameters on the controller
- ✓ Energy of the pendulum when inverted
- ✓ Control gains required to inverted the pendulum automatically

Section 1: State-Space Modeling of the Pendulum Module

1.1 Theory and Background

The rotary pendulum model is shown in Figure 1-1. The rotary arm pivot is attached to the Controls Board system and is actuated. The arm has a length of L_r, a moment of inertia of J_r, and its angle θ increases positively when it rotates counter-clockwise (CCW). The servo (and thus the arm) should turn in the CCW direction when the control voltage is positive ($V_m > 0$).

The pendulum link is connected to the end of the rotary arm. It has a total length of L_p and it center of mass is at $L_p / 2$. The moment of inertia about its center of mass is J_p, which was experimentally determined in the previous section. The inverted pendulum angle α is zero when it is hanging downward and increases positively when rotated CCW.

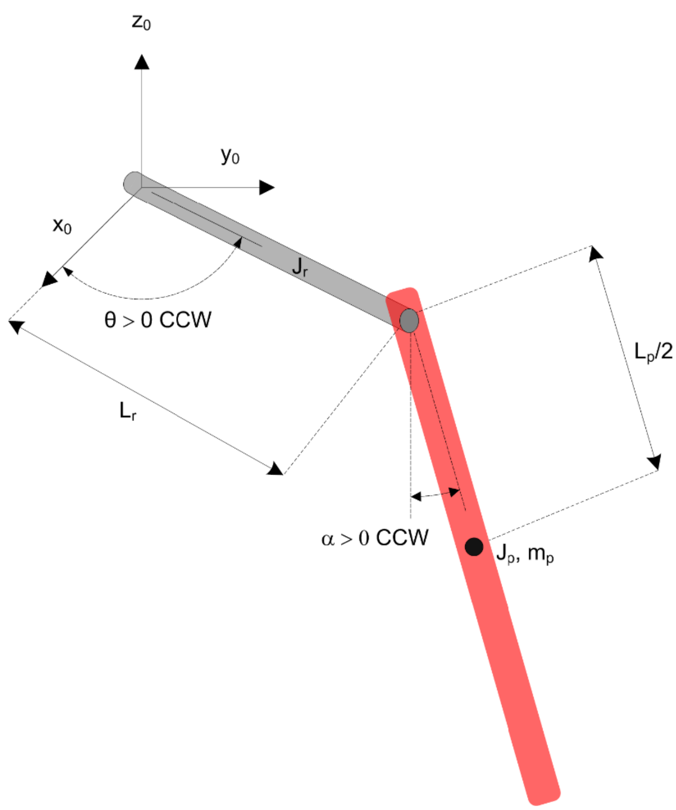


Figure 1-1: Rotary inverted pendulum model

The equations of motion (EOM) for the pendulum system were developed using the Euler-Lagrange method. This systematic method is often used to model complicated systems such as robot manipulators with multiple joints. The total kinetic and potential energy of the system is obtained, then the Lagrangian can be found. A number of derivatives are then computed to yield the EOMs.

The resultant nonlinear EOM are:

Equation 1-1

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 - \frac{1}{4} m_p L_p^2 \, cos(\alpha)^2 + J_r\right) \ddot{\theta} + \left(\frac{1}{2} m_p L_p L_r \, cos(\alpha)\right) \ddot{\alpha} + \left(\frac{1}{2} m_p L_p^2 \, sin(\alpha) \, cos(\alpha)\right) \dot{\theta} \dot{\alpha} - \left(\frac{1}{2} m_p L_p L_r \, sin(\alpha)\right) \dot{\alpha}^2 = \tau - D_r \dot{\theta}$$

and:

Equation 1-2

$$\frac{1}{2} m_p L_p L_r \, cos(\alpha) \, \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} - \frac{1}{4} m_p L_p^2 \, cos(\alpha) \, sin(\alpha) \, \dot{\theta}^2 + \frac{1}{2} m_p L_p g sin(\alpha) = -D_p \dot{\alpha}$$

with an applied torque at the base of the rotary arm generated by the servo motor as described by the equation:

Equation 1-3

$$\tau = \frac{k_m \left(V_m - k_m \dot{\theta}\right)}{R_m}$$

When the nonlinear EOM are linearized about the operating point, the resultant linear EOM for the inverted pendulum are defined as:

Equation 1-4

$$\left(m_p L_r^2 + J_r\right)\ddot{\theta} + \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - D_r \ddot{\theta}$$

and:

Equation 1-5

$$\frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right) \ddot{\alpha} + \frac{1}{2} m_p L_p g \alpha = -D_p \dot{\alpha}$$

Solving for acceleration terms yields:

Equation 1-6

$$\ddot{\theta} = \frac{1}{J_T}\left(-\left(J_p + \frac{1}{4} m_p L_p^2\right) D_r \dot{\theta} + \frac{1}{2} m_p L_p L_r D_p \dot{\alpha} + \frac{1}{4} m_p^2 L_p^2 L_r g \alpha + \left(J_p + \frac{1}{4} m_p L_p^2\right) \tau\right)$$

and:

Equation 1-7

$$\ddot{\alpha} = \frac{1}{J_T}\left(\frac{1}{2} m_p L_p L_r D_r \dot{\theta} - \left(J_r + m_p L_r^2\right) D_p \dot{\alpha} - \frac{1}{2} m_p L_p g \left(J_r + m_p L_r^2\right) \alpha - \frac{1}{2} m_p L_p L_r \tau\right)$$

where:

Equation 1-8

$$J_T = J_p m_p L_r^2 + J_r J_p + \frac{1}{4} J_r m_p L_p^2$$

Linear State-Space Model

The linear state-space equations are:

Equation 1-9

$$\dot{x} = Ax + Bu$$

and:

Equation 1-10

$$y = Cx + Du$$

where *x* is the state, *u* is the control input, *A*, *B,* *C*, and *D* are state-space matrices. For the rotary pendulum system, the state and output are defined:

Equation 1-11

$$x = [\theta \quad \alpha \quad \dot{\theta} \quad \dot{\alpha}]^T$$

and:

Equation 1-12

$$y = [\theta \quad \alpha]^{\mathrm{T}}$$

Section 2: Inverted Pendulum Control

2.1 Theory and Background

In the previous section, you developed a state-space model of the pendulum module in the downward gantry position. In this section you will design a balance controller for the pendulum module. The controller, once engaged, will balance the pendulum in the upright (inverted) position. As such, you will be deriving a new state-space model which represents the inverted pendulum.

We begin with the definition of a linear state-space model. The standard linear state-space equations are

Equation 2-1

$$\dot{x} = Ax + Bu$$

and

Equation 2-2

$$y = Cx + Du$$

where *x* is the state, *y* is the control output, and *u* is the control input. *A*, *B*, *C*, and *D* are the state-space matrices that represent the system. For the rotary pendulum system, the state and output are defined as

Equation 2-3

$$x^T = [\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}]$$

and

Equation 2-4

$$y^T = [x_1 \ x_2]$$

where θ is the angle of the arm, and α is the angle of the pendulum. In the output equation, only the position of the servo and link angles are being measured. Based on this, the *C* and *D* matrices in the output equation are

Equation 2-5

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and

Equation 2-6

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The velocities of the servo and pendulum angles can be computed in the digital controller using the encoder reading by taking the derivative and filtering the result though a low-pass filter.

The linear state-space model that represents the Quanser Controls Board inverted pendulum system is therefore

Equation 2-7

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 149.23 & -0.0104 & 0 \\ 0 & 261.53 & -0.0103 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 49.72 \\ 49.13 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u.$$

Your model may slightly differ based on the specific model parameters of your particular Quanser Controls Board, but this model should be representative of a general example.

Stability

The stability of a system can be determined from its poles:

- Stable systems have poles only in the left-hand plane.
- Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis.

- Marginally stable systems have one pole on the imaginary axis and the other poles in the left-hand plane.

The poles are the roots of the system's characteristic equation. From the state-space model, the characteristic equation of the system can be found using

Equation 2-8

$$\det(sI - A) = 0$$

where *det()* is the determinant function, *s* is the Laplace operator, and *I* the identity matrix. These are the eigenvalues of the state-space matrix *A*.

Controllability

If the control input *u* of a system can take each state variable, x_i where $i = 1 \ldots n$, from an initial state to a final state then the system is controllable, otherwise it is uncontrollable.

Rank Test: The system is controllable if the rank, or number of non-zero rows when the matrix is in row echelon form, of its controllability matrix

Equation 2-9

$$T = [B \; AB \; A^2B \ldots A^nB]$$

equals the number of states in the system,

$$\text{rank}\,(T) = n.$$

Companion Matrix

If (*A, B*) are controllable and *B* has dimension n x 1, then *A* is similar to a companion matrix. Let the characteristic equation of *A* be

Equation 2-10

$$s^n + a_n s^{n-1} + \cdots + a_1.$$

Then the companion matrices of *A* and *B* are

Equation 2-11

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix}$$

and

Equation 2-12

$$\tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Define a transformation matrix *W* as

$$W = T\tilde{T}^{-1}$$

where *T* is the controllability matrix defined in Equation 2-9 and

$$\tilde{T} = \left[\tilde{B}\ \tilde{B}\tilde{A} \ldots\ \tilde{B}\tilde{A}^n\right].$$

Then by applying the standard state transformation definition, we can show that

$$W^{-1}AW = \tilde{A}$$

and

$$W^{-1}B = \tilde{B}.$$

Pole Placement

If (A,B) are controllable, then pole placement can be used to design the controller. Given the control law $u = -Kx$, the state-space in Equation 2-1 becomes

$$\begin{aligned}\dot{x} &= Ax + B(-Kx)\\ &= (A - BK)x\end{aligned}$$

To illustrate how to design gain K, consider the following system

Equation 2-13

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -1 & -5 \end{bmatrix}$$

and

Equation 2-14

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that A and B are already in the companion form. We want the closed-loop poles to be at *[-1 -2 -3]*. The desired characteristic equation is therefore

Equation 2-15

$$(s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

For the gain $K = [k1\ k2\ k3]$, apply control $u = -Kx$ and get

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 - k_1 & -1 - k_2 & -5 - k_3 \end{bmatrix}.$$

The characteristic equation of $A - KB$ is

Equation 2-16

$$s^3 + (k_3 + 5)s^2 + (k_2 + 1)s + (k_1 - 3)$$

Equating the coefficients between Equation 2-16 and the desired polynomial in Equation 2-15

$$k_1 - 3 = 6$$
$$k_2 + 1 = 11$$
$$k_3 + 5 = 6$$

Solving for the gains, we find that a gain of $K = [9\ 10\ 1]$ is required to move the poles to their desired location. We can generalize the procedure to design a gain *K* for a controllable *(A,B)* system as follows:

Step 1 Find the companion matrices $\tilde{A}$ and $\tilde{B}$. Compute $W = T\tilde{T}^{-1}$.

Step 2 Compute $\widetilde{K}$ such that the poles of $\tilde{A} - \tilde{B}\widetilde{K}$ are assigned to the desired locations. Applying the control law $u = -Kx$ to the dgeneral system given in Equation 2-11,

$$\tilde{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{bmatrix}$$

Step 3 Find $K = \widetilde{K}W^{-1}$to get the feedback gain for the original system *(A, B)*.

Remark It is important to do the $\widetilde{K} \to K$ conversion. Remember that *(A, B)* represents the actual system while the companion matrices $\tilde{A}$ and $\tilde{B}$ do not.

Desired Poles

The rotary inverted pendulum system has four poles. As depicted in Figure 2-1, poles p_1 and p_2 are the complex conjugate dominant poles and are chosen to satisfy the natural frequency, ω_n, and damping ratio, ζ, specifications. If we let the conjugate poles be

Equation 2-18

$$p_1 = -\sigma + j\omega_d$$

and

Equation 2-19

$$p_2 = -\sigma - j\omega_d$$

where $\sigma = \zeta\omega_n$, j is the imaginary unit, and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. The remaining closed-loop poles, p_3, and p_4, are placed along the real-axis to the left of the dominant poles, as shown in Figure 2-1.

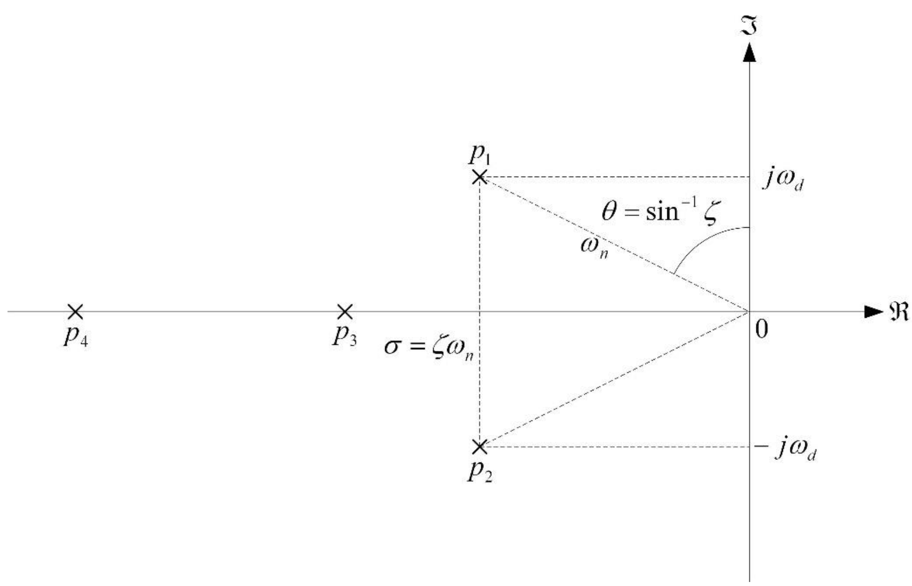


Figure 2-1: Desired closed-loop pole locations

Feedback Control

The feedback control loop that balances the rotary pendulum is illustrated in Figure 2-2. The reference state is defined

$$x_d = [\theta_d \quad 0 \quad 0 \quad 0]$$

where θ_d is the desired rotary arm angle. The controller is

$$u = K(x_d - x).$$

Note that if $x_d = 0$ then $u = -Kx$, which is the control algorithm used in the pole-placement algorithm.

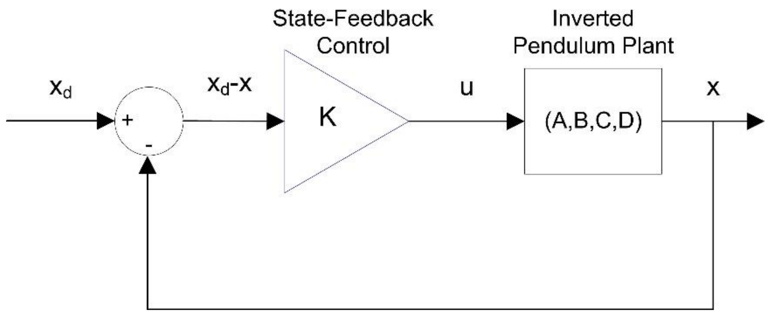


Figure 2-2: State-feedback control loop

2.2 Implement

Control Design

1. The open-loop poles of the inverted pendulum are located at -16.17, 16.17, -0.005, and 0. Using the open-loop poles, find the characteristic equation of *A*.
2. Find the corresponding companion matrices $\tilde{A}$ and $\tilde{B}$.
3. Find the location of the two dominant poles, p_1 and p_2, based on the following specifications
   - ζ = **0.7**
   - ω_n = **4** rad/s

4. Give the desired characteristic equation if the other poles are placed at p_3 = -30 and p_4 = -40.
5. When applying the control $u = -\tilde{K}x$ to the companion form, it changes $(\tilde{A}, \tilde{B})$ to $(\tilde{A} - \tilde{B}\tilde{K}, \tilde{B})$. Find the gain $\tilde{K}$ that assigns the poles to their new desired location.
6. Open the project **Quanser Controls Board.lvproj**, and open **Balance Control Design.vi**.

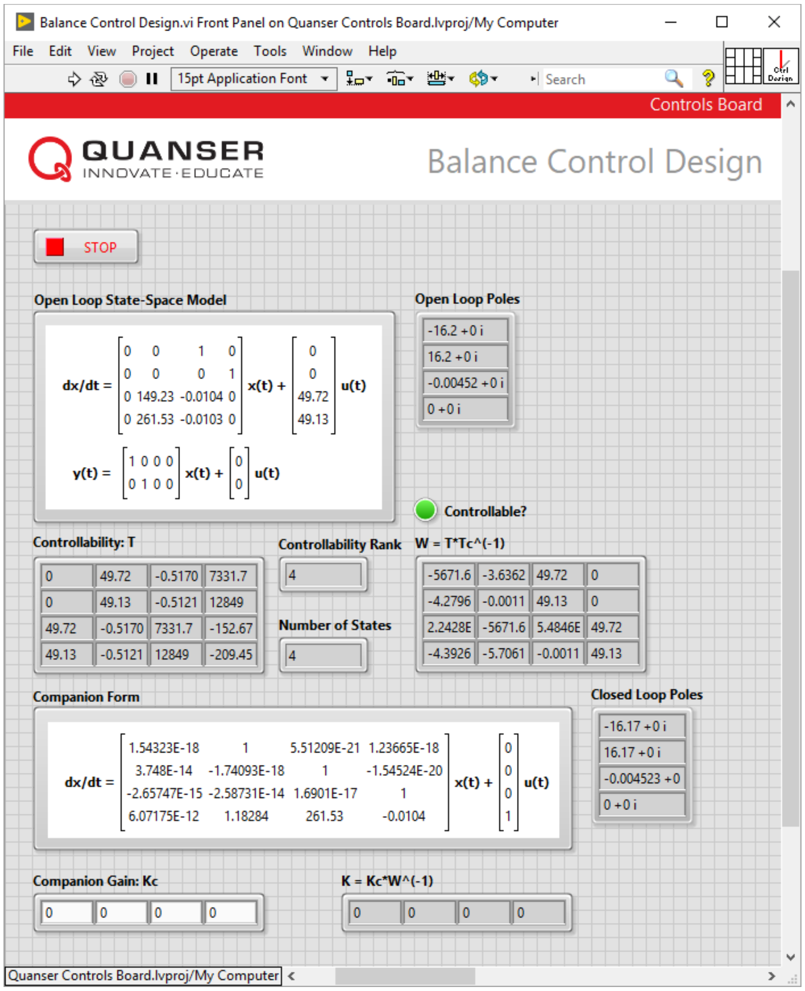
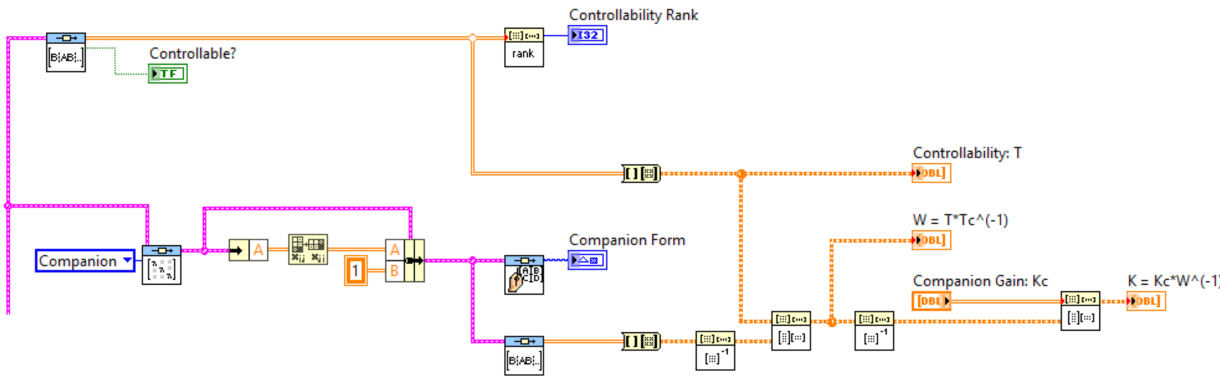


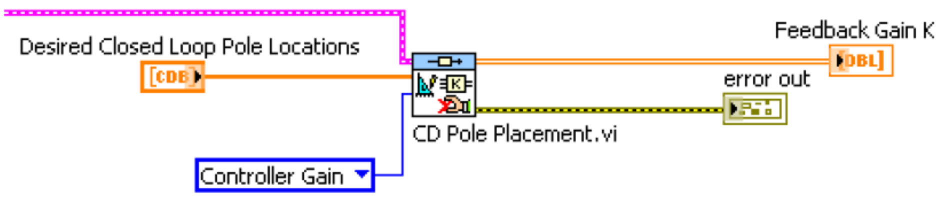
Figure2-3: Application for balance control design

7. Run the VI. The model should match the model given in Equation 2-7.
8. The companion matrices $\tilde{A}$ and $\tilde{B}$ are automatically found. In order to determine the appropriate gain *K*, the transformation matrix must be found. Open the block diagram, and locate the code segment used to calculate the controllability matrix *T*, the companion controllability matrix *Tc*, and the inverse of *Tc* shown below.

Inspect the code segment in order to understand the functionality of the algorithm.



9. Enter the companion gain $\tilde{K}$ that you found in Step 5 into the **Companion Gain: Kc** input on the front panel. Record the resultant feedback gain *K*.
10. Record the closed-loop poles of the system when using the gain *K*.
    Note: The code that is used to determine the required matrices and feedback gain above can be replaced by a single operation shown below. This approach is a much simpler approach to implementing pole placement when using platforms that support common control algorithms.



11. Stop the VI by clicking on the **Stop** button.

Simulation

12. Open the project **Quanser Controls Board.lvproj**, and open **Balance Control Simulation.vi**.

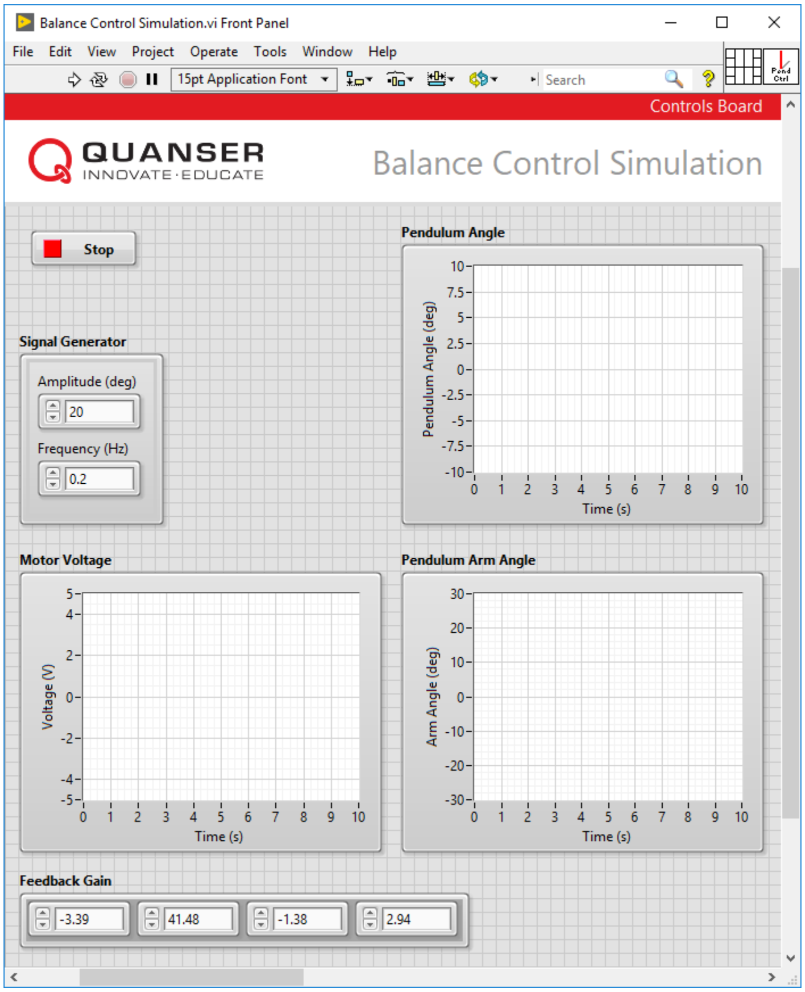


Figure 2-4: Application for Inverted Pendulum Control Simulation

13. Enter the gain that you found in the previous section into the **Feedback Gain** input.
14. Record the simulated response of your designed gain.
15. Given the additional implementation specifications:
    - Maximum pendulum deflection $|\alpha| <$ **15**°.
    - Maximum control effort $|V_m| <$ **5** V.

    Measure the simulated response and determine if the additional specifications listed are met.
16. Stop the VI by clicking on the **Stop** button.

Inverted Pendulum Control

17. Open the project **Quanser Controls Board.lvproj**, and open **Balance Control.vi**.

18. Enter the feedback gain, *K*, from the previous section.
19. Ensure the pendulum is in the hanging down position and is motionless. Run the VI and manually bring up the pendulum to its upright, vertical position. You should feel the voltage kick-in when it is within the range where the balance control engages.
20. Once it is balanced, introduce a ±20 degree rotary arm command by setting **Amplitude (deg)** to 20 in the VI. The response should look similar to your simulation.
21. Record the measured rotary pendulum responses.
22. Measure the maximum pendulum deflection and voltage used. Are the specifications given in Step 15 satisfied for the implementation?
23. Stop the VI by clicking on the **Stop** button.

Section 3: Optimal Control of an Inverted Pendulum

3.1 Theory and Background

Linear Quadratic Regulator (LQR) theory is a technique that is ideally suited for finding the optimal parameters of the pendulum balance controller. Given that the equations of motion of the system can be described in the form

$$\dot{x} = Ax + Bu$$

where *A* and *B* are the state and input system matrices, respectively, the LQR algorithm computes a control law *u* such that the performance criterion or cost function

Equation 3-1

$$J = \int_0^\infty \left(x_{ref} - x(t)\right)^T Q \left(x_{ref} - x(t)\right) + u(t)^T R u(t) dt$$

is minimized. The design matrices *Q* and *R* hold the penalties on the deviations of state variables from their set-point and the control actions, respectively. When an element of *Q* is increased, therefore, the cost function increases the penalty associated with any deviations from the desired set-point of that state variable, and thus the specific control gain will be larger. When the values of the *R* matrix are increased, a larger penalty is applied to the aggressiveness of the control action, and the control gains are uniformly decreased.

In our case the state vector *x* is defined

Equation 3-2

$$x = \left[\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}\right]^T$$

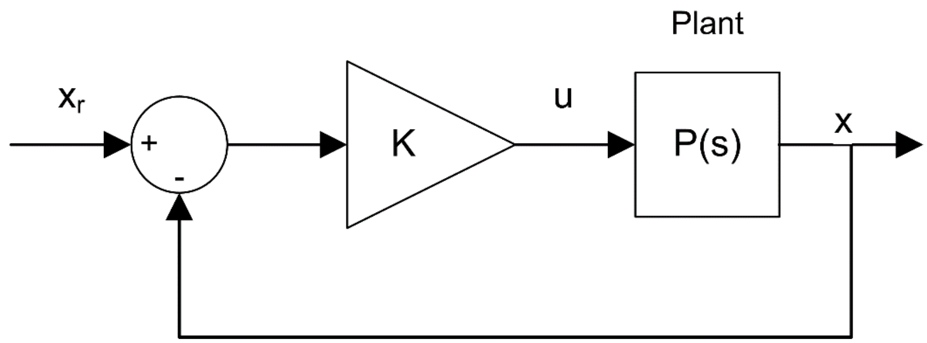


Figure 3-1: Block diagram of balance state-feedback control for rotary pendulum

Since there is only one control variable, *R* is a scalar. The reference signal x_{ref} is set to $[\theta_r\ 0\ 0\ 0]^T$, and the control strategy used to minimize the cost function *J* is thus given by

Equation 3-3

$$u = K\left(x_{ref} - x\right) = k_{p,\theta}(\theta_r - \theta) - k_{p,\alpha}\alpha - k_{d,\theta}\dot{\theta} - k_{d,\alpha}\dot{\alpha}.$$

This control law is state-feedback control, and is illustrated in Figure 3-1. It is equivalent to PD control.

3.2 Implement

LQR design theory has built in support in LabVIEW™ using the Control Design & Simulation module. Given a model of the system in state-space form (with system matrices *A* and *B*) and the weighting matrices *Q* and *R*, the LQR function in the Control Design Toolkit automatically minimizes the cost function Equation 3-1 and computes the optimal feedback control gain automatically.

LQR Control Design

In this experiment, the state-space model is already available. Therefore, the effect of changing the *Q* weighting matrix while *R* is fixed to 1 on the cost function *J* will be explored.

1. Open the project **Quanser Controls Board.lvproj**, and open **Optimal Control Design.vi**.

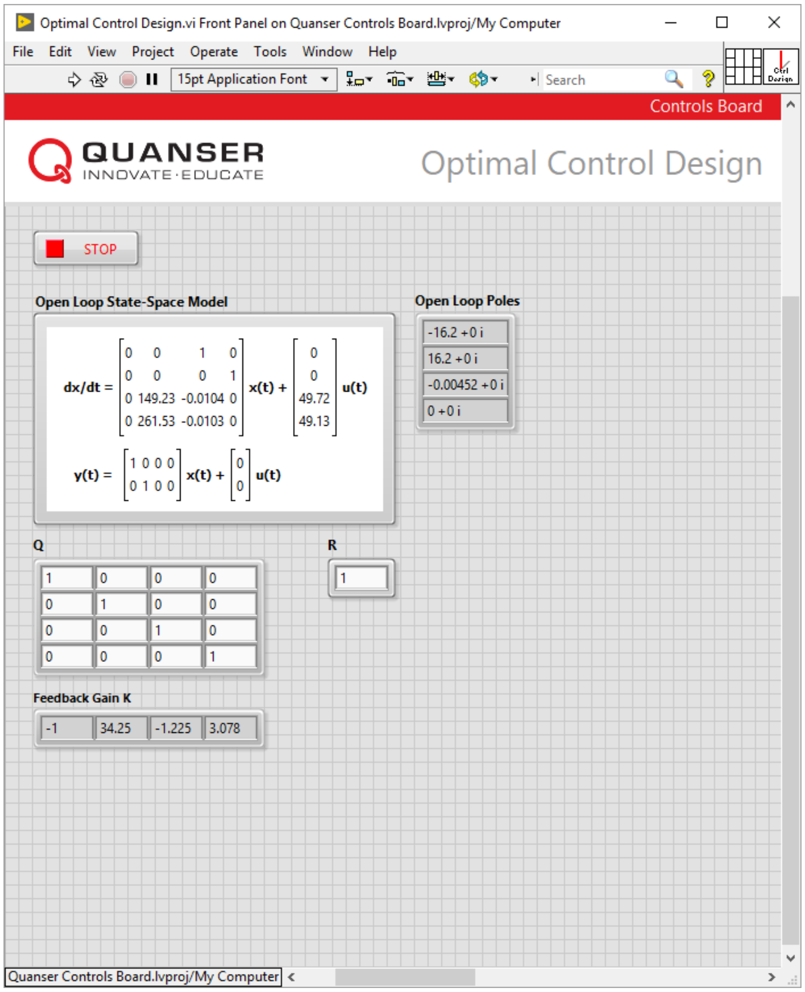


Figure 3-2: VI used to design the balance controller using LQR

2. Ensure that the weighting matrices are set to

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } R = 1.$$

3. Record the gain K that is generated by the VI.
4. Stop the VI, and repeat the process using the following weighting matrices

$$Q = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } R = 1.$$

5. Record the new gain.
6. Stop the VI by clicking on the **Stop** button.

LQR Balance Control

7. Once again open the **Balance Control.vi** shown in Figure 3-2.

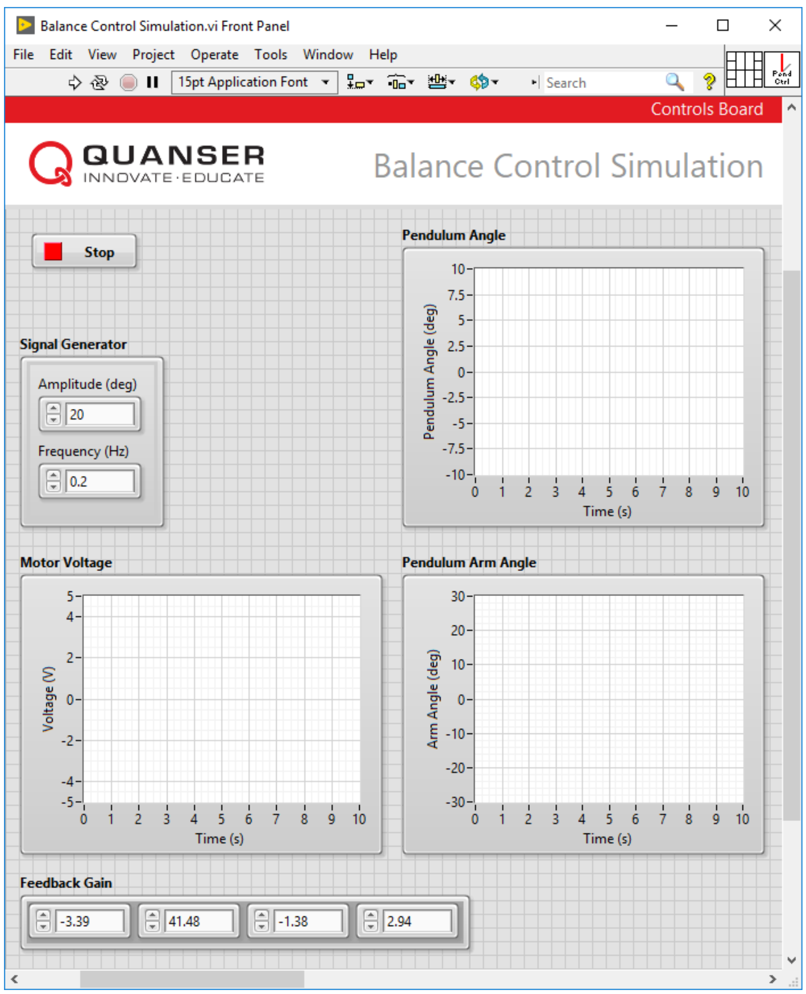


Figure 3-3: VI used to balance the pendulum using LQR

8. Make sure that the feedback gains are set to those recorded in Step 3 of the LQR Control Design section.
9. Run the VI.
10. Manually rotate the pendulum in the upright position until the controller engages.
11. Once the pendulum is balanced, set the **Amplitude (deg)** control to 30 to make the arm angle go between ±30°, and set the **Frequency (Hz)** control to 0.1.
12. The scopes should look similar to those shown in Figure 3-3. Record the response of the rotary arm, pendulum, and controller voltage.

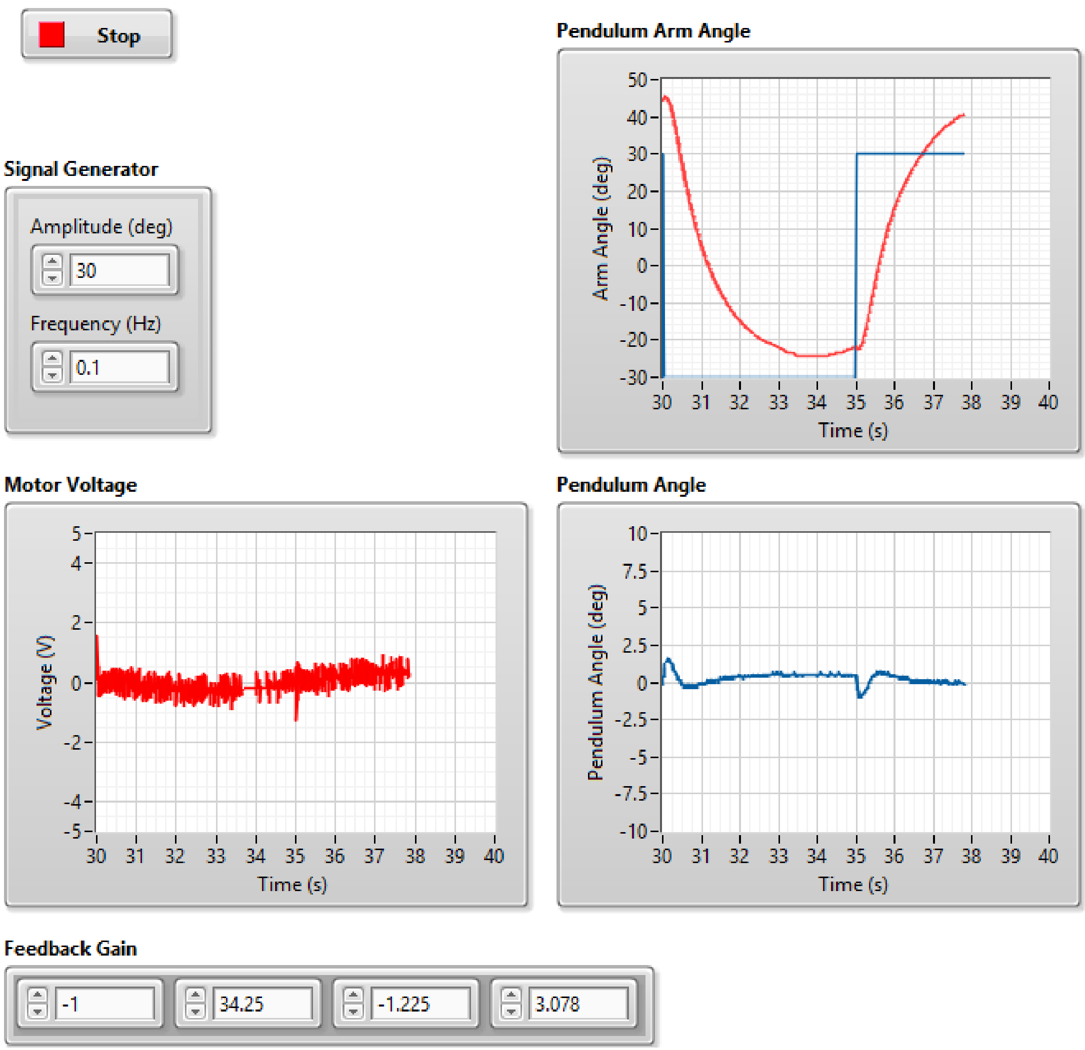


Figure 3-4: Rotary pendulum response

13. Stop the VI. Update the feedback gains with the values in Step 5 of the LQR Control Design section and repeat your analysis.

14. Finally, using **Optimal Control Design.vi** adjust the value of *R* to 0.8 and record a new set of feedback gains. Update the feedback gains in the balance control VI with these new gains, and observe and record the resultant gain and response.
15. Use **Optimal Control Design.vi** to generate a set of new feedback gains to meet the following maximum control specifications:
    - Pendulum deflection: ±5°
    - Overshoot: 20%
    - Peak time: 0.8 s

    Describe your experimental procedure to find the necessary control gain.
    Note: Assume that the value of *R* can remain set to **0.8** during tuning.
16. List the resulting LQR *Q* matrix and control gain *K* used to yield the desired results. Record the responses using this new control gain and briefly outline how the response changed.
17. Stop the VI by clicking on the **Stop** button.

Section 4: Swing-Up Hybrid Control

4.1 Theory and Background

Energy Control

In theory, if the arm angle of the pendulum system is kept constant and the pendulum is given an initial perturbation, the pendulum will keep on swinging with constant amplitude. The idea of energy control is based on the preservation of energy in ideal systems: The sum of kinetic and potential energy is constant. However, friction will damp the oscillation in practice and the overall system energy will not be constant. It is possible to measure the loss of energy with respect to the pivot acceleration, which in turn can be used to find a controller to swing up the pendulum.

The dynamics of the pendulum can be redefined in terms of the pivot acceleration, *u*, as

Equation 4-1

$$J_p\ddot{\alpha} + \frac{1}{2}M_p g L_p \sin\alpha = \frac{1}{2}M_p L_p u \cos\alpha.$$

Here, *u* is the linear acceleration of the pendulum.

The potential energy of the pendulum is

$$E_p = \frac{1}{2}M_p g L_p(1 - \cos\alpha),$$

and the kinetic energy is

$$E_k = \frac{1}{2}J_p\dot{\alpha}^2.$$

The pendulum angle, *α*, and the lengths of the pendulum are illustrated in the free body diagram in Figure 4-1.

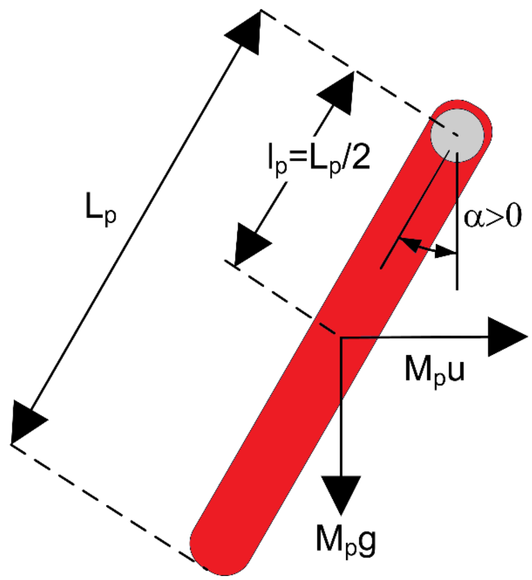


Figure 4-1: Rotary pendulum response

The potential energy is zero when the pendulum is at rest at $\alpha = 0$ and equals $M_p g L_p$ when the pendulum is upright at $\alpha = \pm\pi$. The sum of the potential and kinetic energy of the pendulum is

Equation 4-2

$$E = \frac{1}{2} J_p \dot{\alpha}^2 + \frac{1}{2} M_p g L_p (1 - \cos\alpha).$$

Differentiating Equation 4-2 yields

Equation 4-3

$$\dot{E} = \dot{\alpha}\left(J_p \ddot{\alpha} + \frac{1}{2} M_p g L_p \sin\alpha\right).$$

Using Equation 4-1, the terms can be rearranged as

$$J_p \ddot{\alpha} = -M_p g l_p \sin\alpha + M_p u l_p \cos\alpha$$

which eventually yields

$$\dot{E} = M_p u l_p \dot{\alpha} \cos \alpha.$$

Since the acceleration of the pivot is proportional to current driving the arm motor and thus also proportional to the drive voltage, it is possible to control the energy of the pendulum with the proportional control law

Equation 4-4

$$u = (E_r - E)\dot{\alpha} \cos \alpha.$$

By setting the reference energy to the pendulum potential energy ($E_r = E_p$), the control law will swing the link to its upright position. Notice that the control law is nonlinear because the proportional gain depends on the cosine of the pendulum angle *α*. Further, the control changes sign when $\dot{\alpha}$ changes sign and when the angle is ±90 degrees.

For the system energy to change quickly, the magnitude of the control signal must be large. As a result the following swing-up controller is implemented in the controller as

Equation 4-5

$$u = \mathrm{sat}_{u_{\max}}\left(\mu(E_r - E)\mathrm{sign}(\dot{\alpha} \cos \alpha)\right)$$

where μ is a tunable control gain and the $\mathrm{sat}_{u\max}$ function saturates the control signal at the maximum acceleration of the pendulum pivot,$u_{\max}$. The expression $\mathrm{sign}(\dot{\alpha}\cos\alpha)$ is used to enable faster control switching.

Hybrid Control

The energy swing-up control in Equation 4-4 (or Equation 4-5) can be combined with the balancing control law from the Balance Control Lab to obtain a control law that swings up the pendulum and then balances it.

Similarly, the balance control is to be enabled when the pendulum is within ±20 degrees. When it is not enabled, the swing-up control is engaged. Thus the switching can be described mathematically by

Equation 4-6

$$u = \begin{cases} u_{\text{bal}} & \text{if } |\alpha| - \pi \le 0.35 \text{ rad} \\ u_{\text{swing}} & \text{otherwise} \end{cases}$$

4.2 Implement

The VI shown below implements the algorithm described in Section 4.1 to swing-up and balance the pendulum.

Energy Control

1. Open the project **Quanser Controls Board.lvproj**, and open **Swing-up Control.vi**.

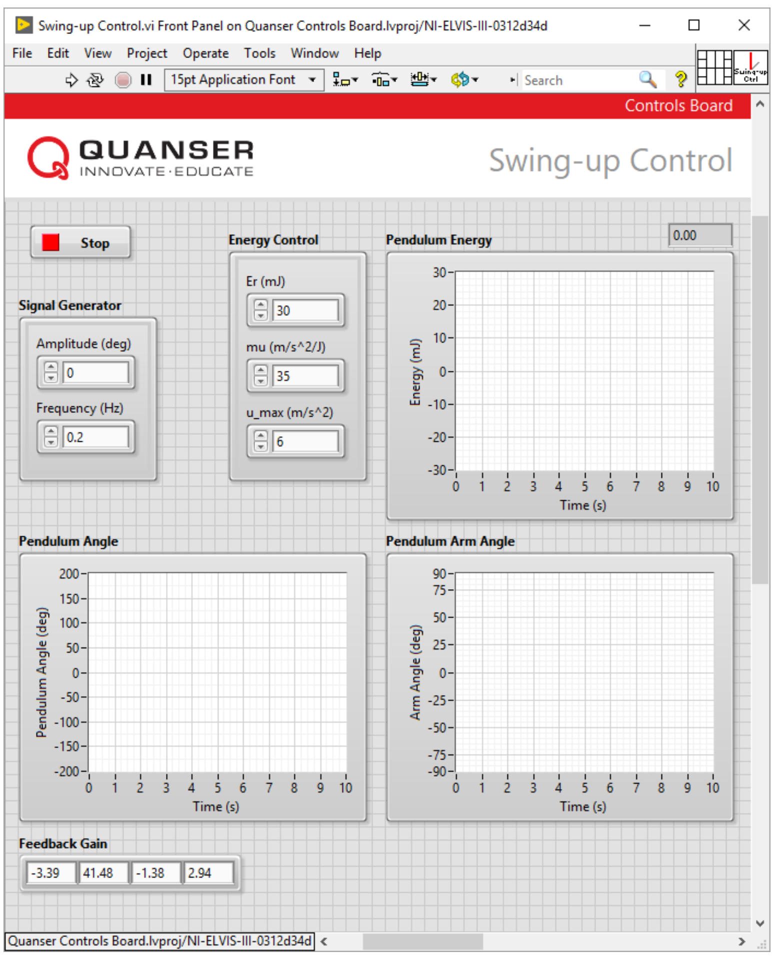


Figure 4-2: Application for hybrid swing-up control

2. Ensure that the swing-up controller is disabled by setting the gain, μ, to 0.
3. Run the VI.
4. Manually rotate the pendulum to different angles and examine the pendulum energy shown in the **Pendulum Angle (deg)** and **Pendulum Energy (mJ)** charts.
5. Record the energy when the pendulum is being balanced upright.
6. Click on the **Stop** button to return the pendulum to the downward gantry position.

Hybrid Swing-Up Control

7. Set the swing-up control parameters to the following values:
    - Er (mJ) = **10**
    - mu (m/s^2/J) = **50**
    - u_max (m/s^2) = **6**
8. Run the VI
9. If the pendulum is not moving, gently perturb the pendulum with your hand from the downward position.
10. Vary the reference energy, E_r, between 10.0 mJ and 20.0 mJ. As it is changed, examine the pendulum angle and energy response in the **Pendulum Angle (deg)** and the **Pendulum Energy (mJ)** charts and the control signal in the **Motor Voltage (V)** chart.
11. Fix the value of E_r at 20.0 mJ and vary the swing-up control gain μ between 20 and 60 m/s^2/J. Observe any changes in the performance of the energy controller.
12. Stop the VI by clicking on the **Stop** button.
13. Set the swing-up control parameters to the following values:
    - mu (m/s^2/J) = **20**
    - u_max (m/s^2) = **6**
14. Based on your observations from the previous section, enter an appropriate value for the reference energy parameter, E_r.
15. Make sure that the pendulum is hanging down motionless and the encoder cable is not interfering with the pendulum.
16. Run the VI.
17. The pendulum should begin going back and forth. If not, perturb the pendulum lightly with your hand. Click on the **Stop** button if the pendulum appears to be unstable.
18. Gradually, in increments of 5 m/s^2/J, increase the swing-up gain μ until the pendulum swings up to the vertical position. Capture the response of the

pendulum, pendulum energy, and motor voltage. Be sure to record the swing-up gain that was required.

19. Stop the VI by clicking on the **Stop** button.

5. For the Report

The report format will be a MEMO. Be sure to include:

Inverted Pendulum Control

1. Based on the number of states and rank of the controllability matrix shown in the Control Design VI, is the system controllable?
2. Record the value of the feedback gain *K* that was calculated in Step 9.
3. Attach the simulated response of your designed gain.
4. Attach the response that was recorded in Step 20.

Optimal Control of an Inverted Pendulum

1. How does changing q11 affect the generated control gain? Based on the description of LQR in the background section, is this what you expected?
2. What is the response of the system in Step 5?
3. What is the response of the system in Step 16?

Swing-Up Hybrid Control

1. What do you notice about the energy of the pendulum when it is moved to various angles? Does the measured energy when the pendulum is balanced upright make sense according to the equations in Section 4.1?
2. Attach the responses specified in Step 10 showing how changing the reference energy affects the system.
3. Attach your responses from Step 18. What gain was required?