FORCED AND FREE VIBRATIONS

LAB 1

FORCED AND FREE VIBRATIONS

- Rayleigh's Energy Method
- Simply Supported Beam
  - Free Vibrations
  - Damped Vibrations
  - Forced Vibrations
- Tunned Mass Damper
- Two DOF System
  - Free Vibration
  - Undamped Vibration Absorber
  - Video Demo-Vibration Absorber
  - Video Demo-Mode 1 & Mode 2

RAYLEIGH'S ENERGY METHOD

- Conservation of energy

$$T_1 + U_1 = T_2 + U_2$$

U_i:potential energy; T_i:kinematic energy

- $U_1 = 0$: reference potential energy, $T_2 = 0$

$$T_{\max} = U_{\max}$$

Rayleigh's improved beam theory accounts for the both the mass of the excitation weight and the mass of the beam, deriving an effective mass in the following form:

$$m_{effective} = m_{exciter} + \frac{17}{35} m_{beam}$$

$$T = \frac{1}{f} = 2\pi \sqrt{\frac{m_{effective} l_{beam}^3}{48 E I_{beam}}}$$

EFFECT OF SPRING MASS

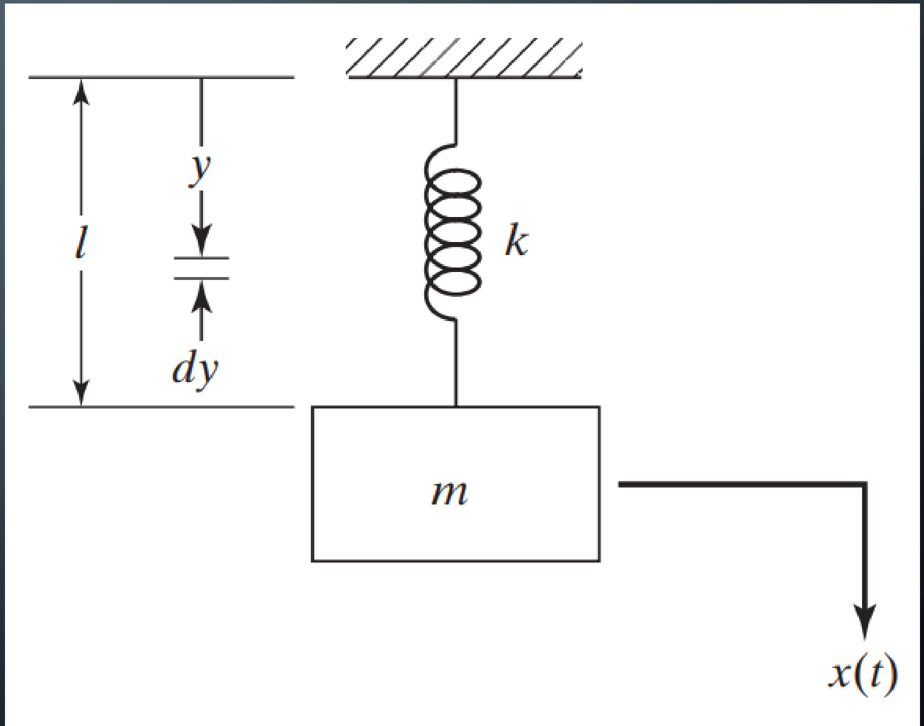
- Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Figure
- Kinetic energy

$$T = \frac{1}{2}m\dot{x}^2 + \int_{y=0}^{l} \frac{1}{2}\left(\frac{m_s}{l}dy\right)\left(\frac{y\dot{x}}{l}\right)^2$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

- Potential energy

$$U = \frac{1}{2}kx^2$$



EFFECT OF SPRING MASS

- Assume harmonic motion

$$x(t) = X \cos \omega_n t$$

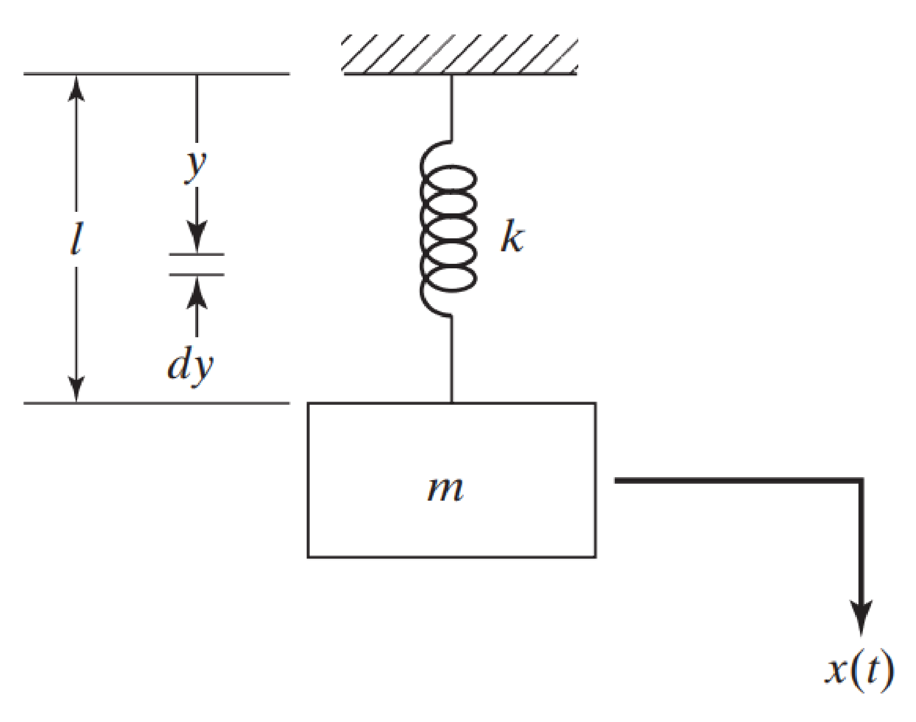
- Maximum energy

$$T_{\max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right) X^2 \omega_n^2$$

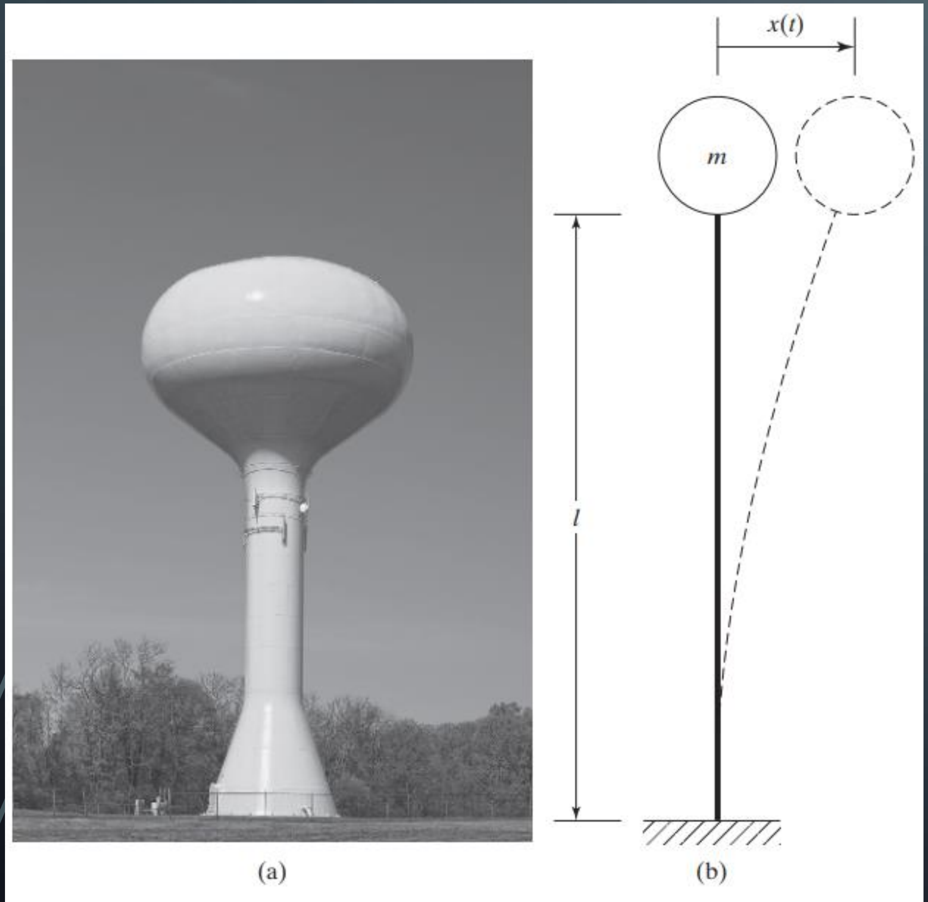
$$U_{\max} = \frac{1}{2} k X^2$$

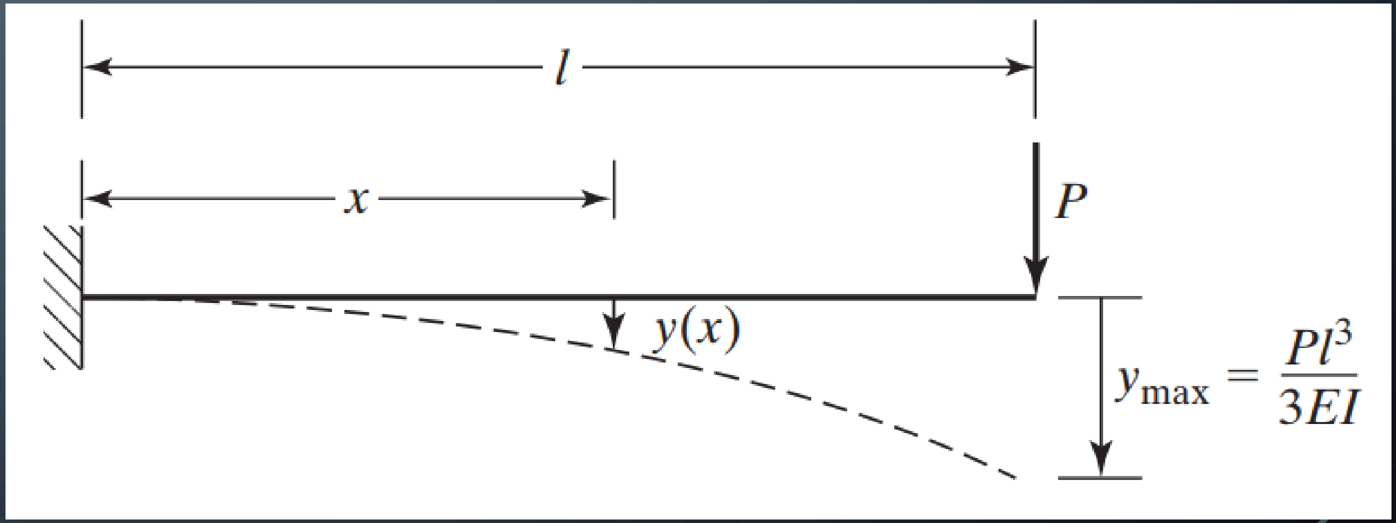
- Natural frequency

$$\omega_n = \left(\frac{k}{m + \frac{m_s}{3}}\right)^{1/2}$$

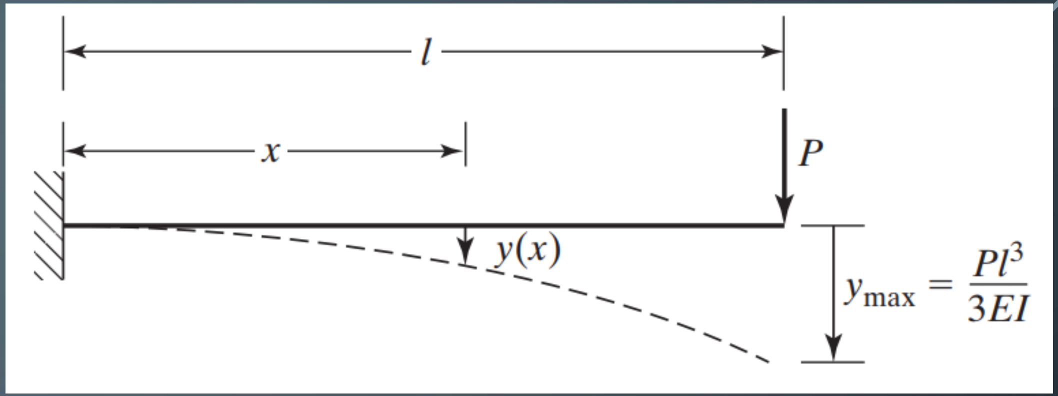


EFFECT OF COLUMN MASS





EFFECT OF COLUMN MASS



- Find the natural frequency of transverse vibration of the water tank including the mass of the column.
- Deflection

$$y(x) = \frac{Px^2}{6EI}(3l - x) = \frac{y_{\max}}{2l^3}(3x^2 l - x^3)$$

- Kinetic energy

$$T_{\max} = \frac{1}{2}\int_0^l \frac{m}{l}\{\dot{y}(x)\}^2 dx$$

$$\dot{y}(x) = \frac{\dot{y}_{\max}}{2l^3}(3x^2 l - x^3)$$

EFFECT OF COLUMN MASS

- Kinetic energy

$$T_{\max} = \frac{m}{2l}\left(\frac{\dot{y}_{\max}}{2l^3}\right)^2 \int_0^l (3x^2 l - x^3)^{\,2} dx$$

$$= \frac{1}{2}\left(\frac{33}{140}m\right)\dot{y}_{\max}^2$$

- Equivalent mass

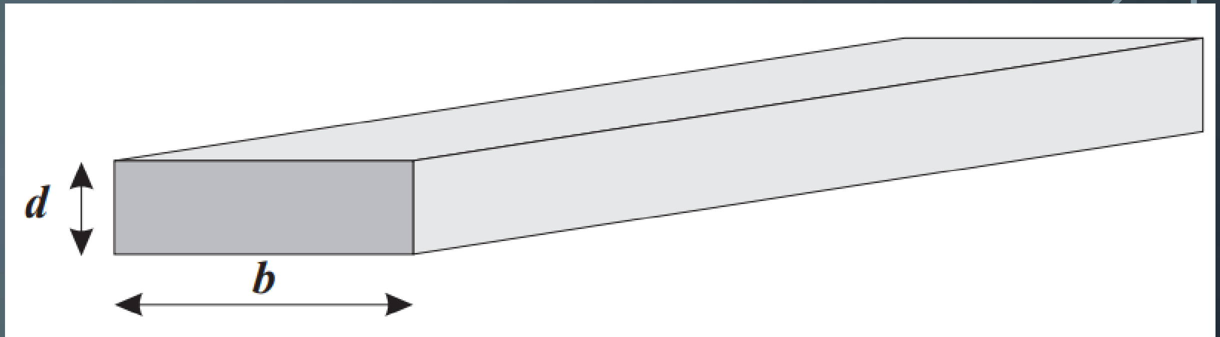
$$T_{\max} = \frac{1}{2} m_{\mathrm{eq}} \dot{y}_{\max}^2$$

$$m_{\mathrm{eq}} = \frac{33}{140} m$$

- Effective mass

$$m_{eff} = M + m_{\mathrm{eq}} = M + \frac{33}{140} m$$

SIMPLY SUPPORTED BEAM

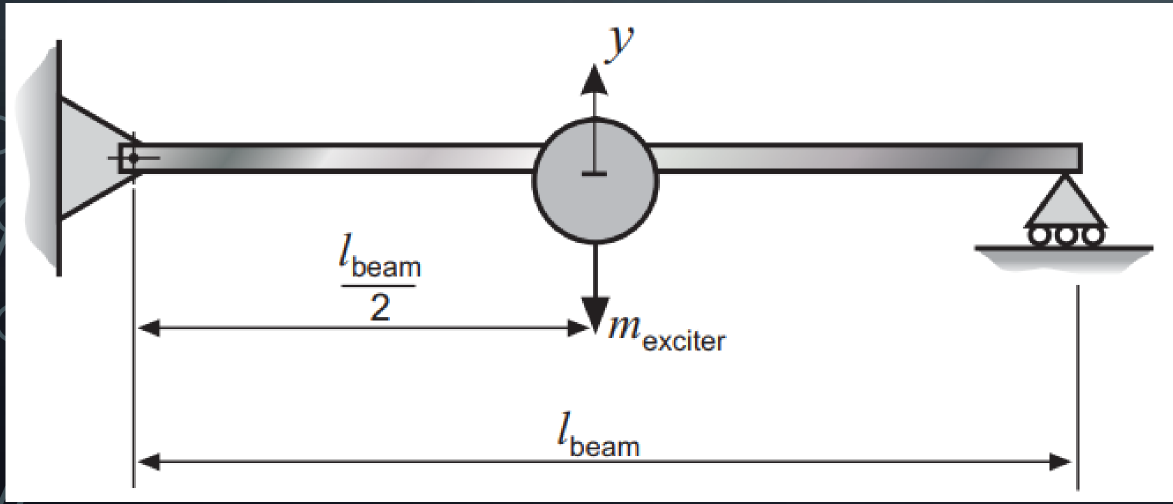


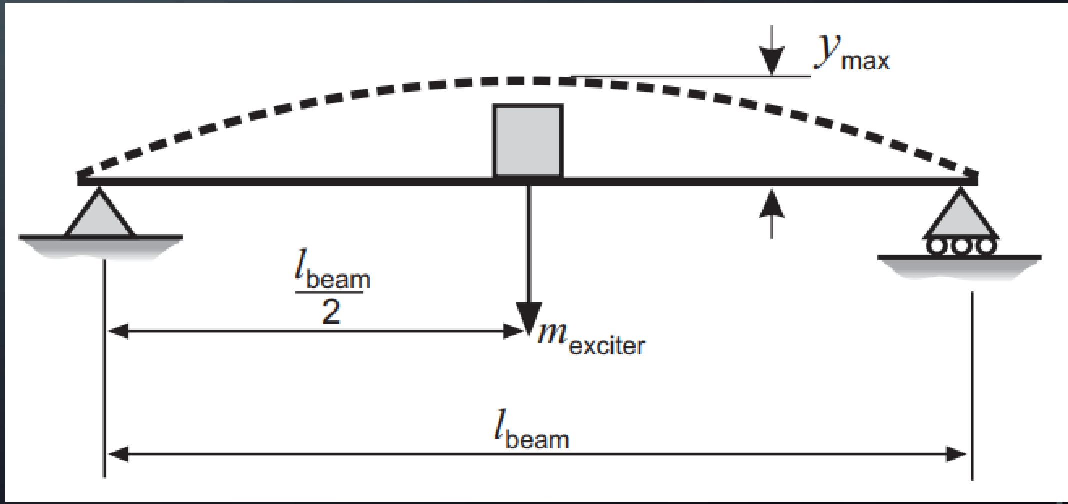
- Static deflection

$$\delta_{st} = \frac{m_{exciter} g l_{beam}^3}{48 E I_{beam}}, I_{beam} = \frac{bd^3}{12}$$

- Flexural rigidity

$$k_{beam} = \frac{48 E I_{beam}}{I_{beam}^3}$$

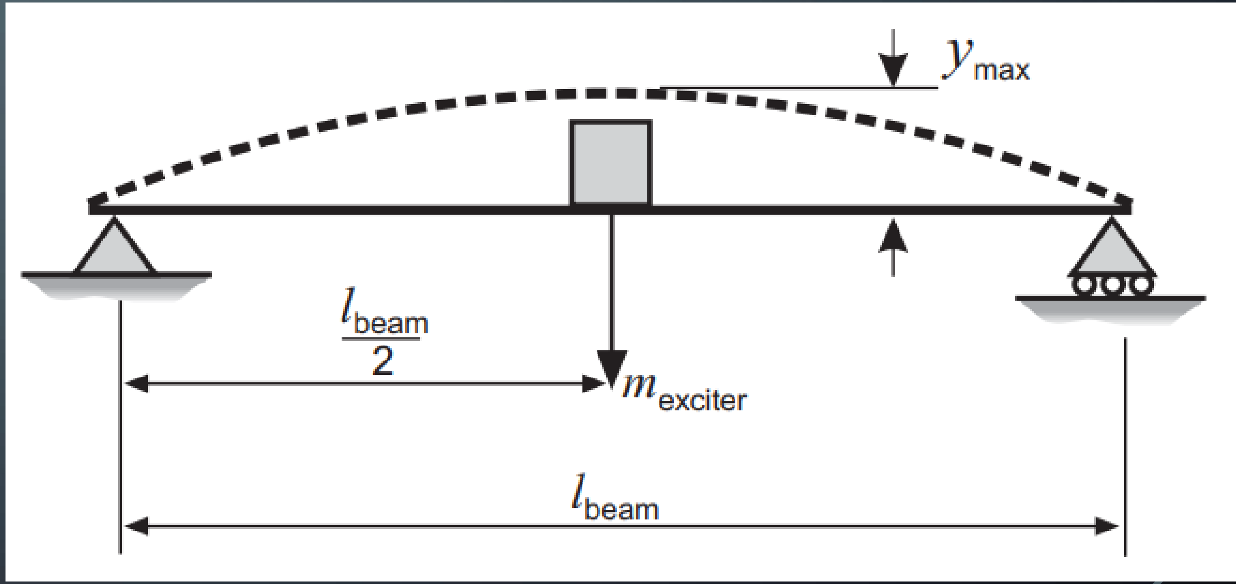




SIMPLY SUPPORTED BEAM FREE VIBRATION

- Beam deflection equation

$$y = y_{max}\left(\frac{3xl_{beam}^2 - 4x^3}{l_{beam}^3}\right) \quad \left\{\frac{x}{l} \leq \frac{1}{2}\right\}$$



- Free vibration

$$m_{exciter}\ddot{y} + k_{beam}y = 0$$

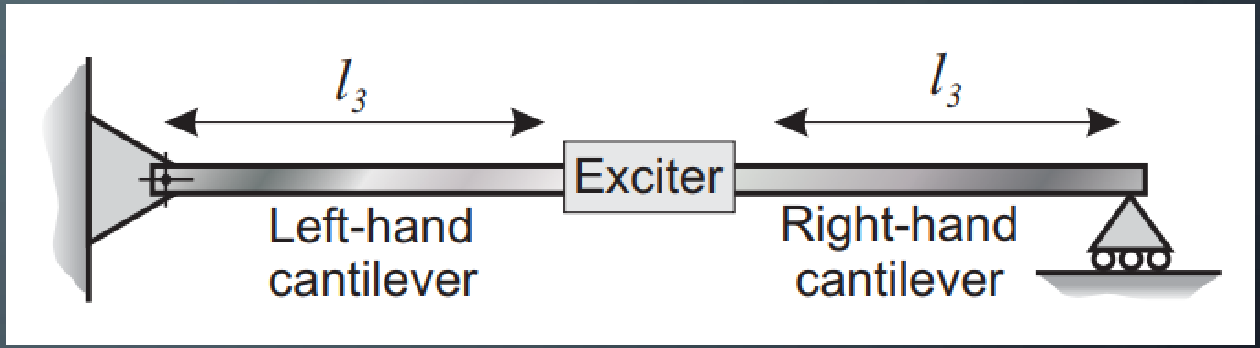
$$\ddot{y} + \omega^2 y = 0$$

$$\omega^2 = \frac{k_{beam}}{m_{exciter}}$$

SIMPLY SUPPORTED BEAM FREE VIBRATION

- Effective mass (Rayleigh)

$$m_{effective} = m_{exciter} + \frac{17}{35} m_{beam}$$



- Corrected rigidity

$$\delta_{st} = \frac{(m_{effective}/2) g l_3^3}{3EI_{beam}}$$

$$= \frac{m_{effective} g l_3^3}{6EI_{beam}}$$

$$k_{beam} = \frac{6EI_{beam}}{l_3^3}$$

$$k_{beam} = \frac{6EI_{beam}}{\frac{l_{beam}}{2}^3}$$

$$T = \frac{1}{f} = 2\pi \sqrt{\frac{m_{effective} \frac{l_{beam}}{2}^3}{6EI_{beam}}}$$

- Improved natural frequency

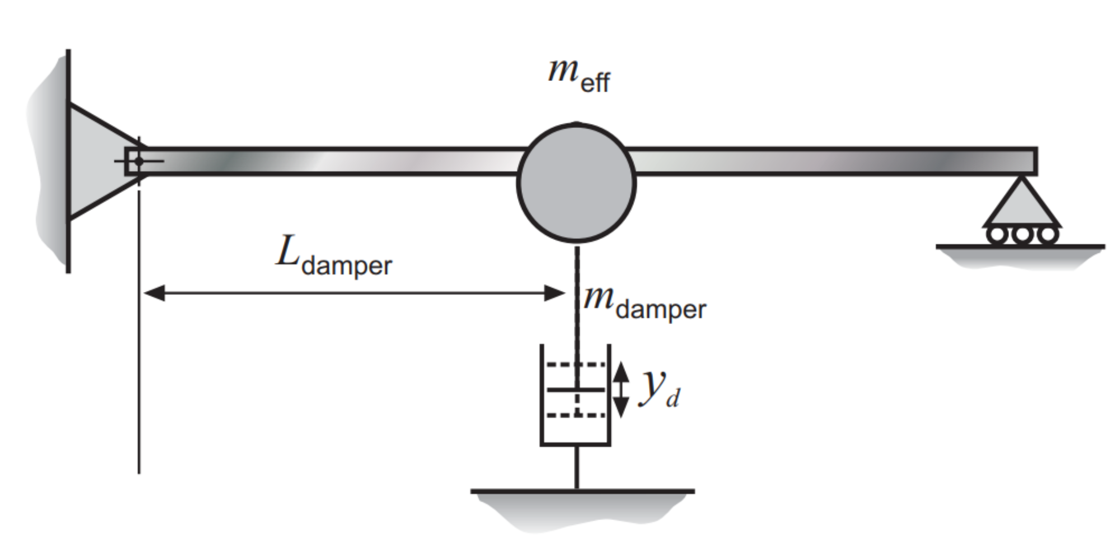
$$f = \frac{1}{2\pi} \sqrt{\frac{6EI_{beam}}{m_{effective} l_3^3}}$$

SIMPLY SUPPORTED BEAM DAMPED VIBRATION

- Damped vibration

$$m_{eff}\ddot{y} + c\dot{y} + k_{eff}y = 0$$

where $m_{eff} = m_{exciter} + \frac{17}{35}m_{beam} + m_{damper}$



- Standard form

$$\ddot{y} + 2\gamma\dot{y} + \omega^2 y = 0$$

$$\gamma = \frac{c}{2m_{eff}}$$

- Solution

$$y = Ae^{-\gamma t}\cos(\omega_d t - \alpha_t)$$

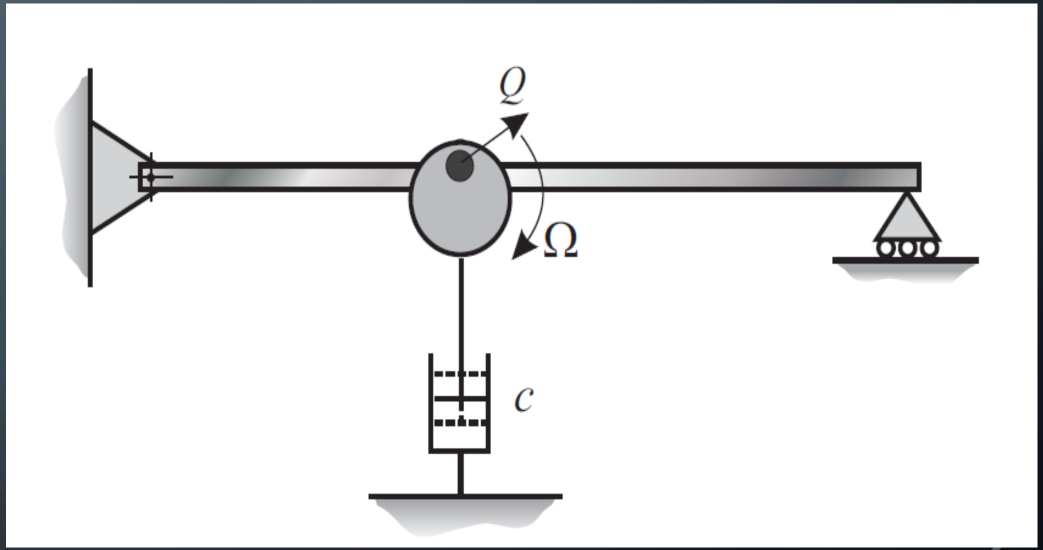
SIMPLY SUPPORTED BEAM FORCED VIBRATIONS

- Equation of motion

$$m_{eff}\ddot{y} + c\dot{y} + k_{eff}y = Q\sin\Omega t$$

$$m_{eff} = m_{mass} + \frac{17}{35}m_{beam} + m_{damper}$$

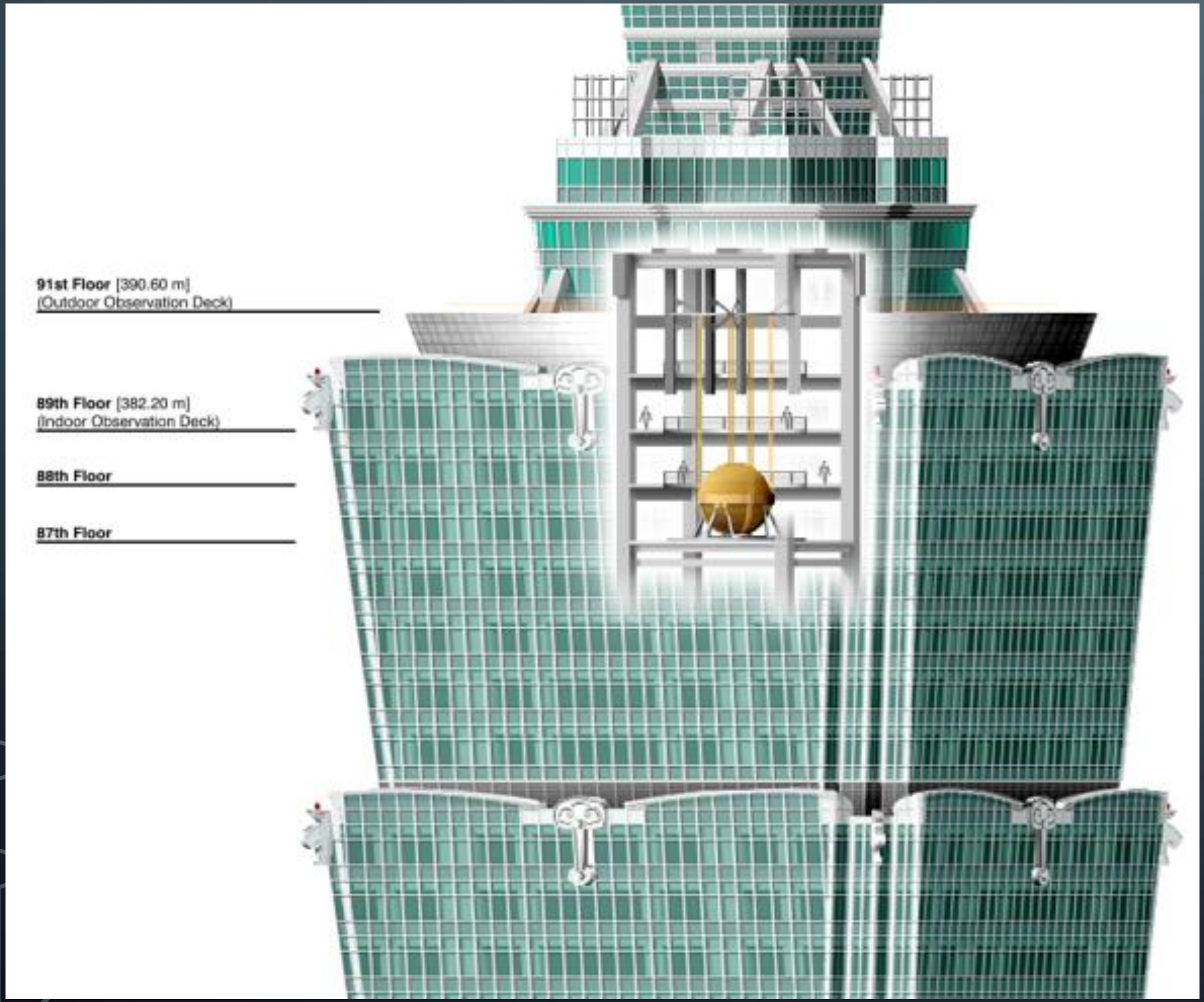
$$\ddot{y} + 2\gamma\dot{y} + \omega^2 y = \frac{Q}{m_{eff}}\sin\Omega t$$

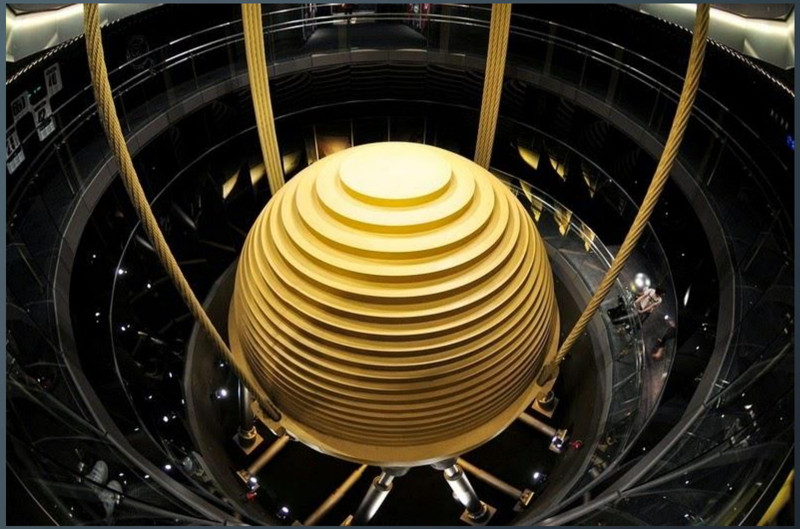


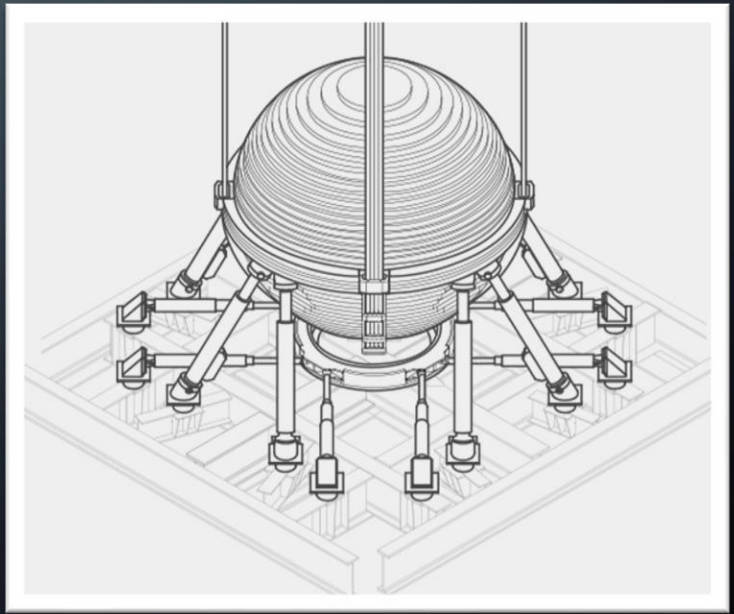
- Solution

$$y = \frac{Q}{k_{eff}}\beta\sin(\Omega t - \alpha)$$

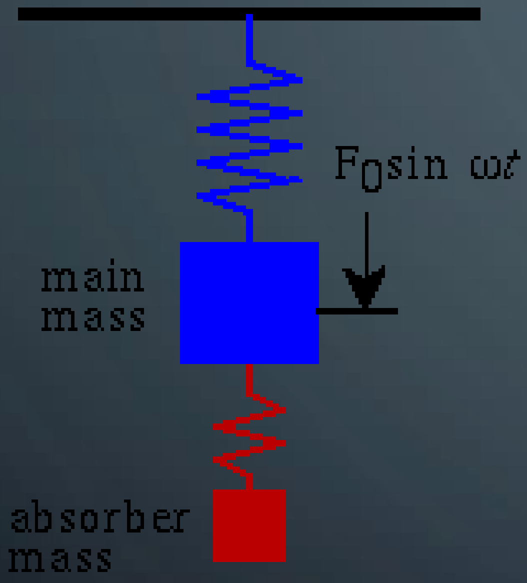
TAIPEI 101-TUNED MASS DAMPER

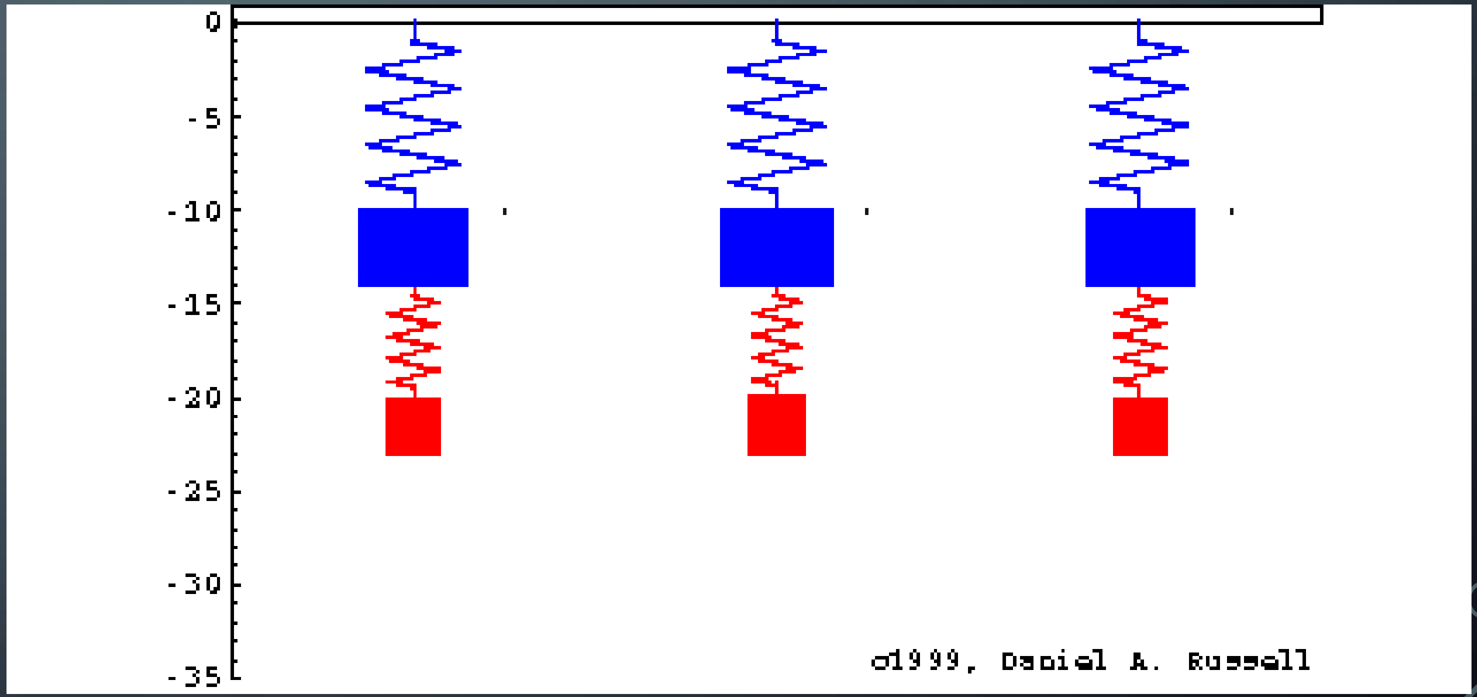




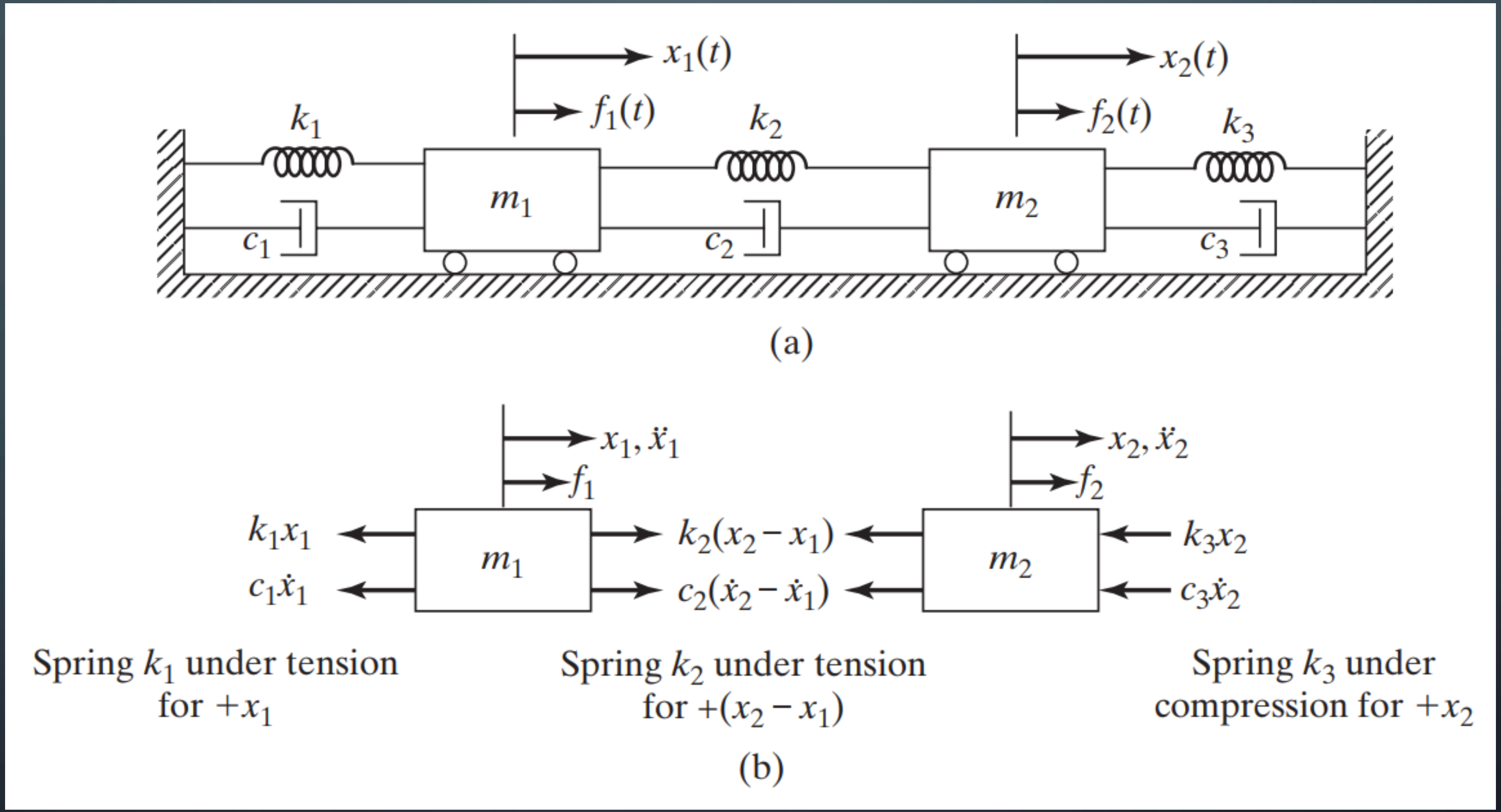


TWO DOF SYSTEM





TWO DOF SYSTEM



$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

TWO DOF SYSTEM

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{f}$$

$$m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad c = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \quad k = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad \vec{f} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

TWO DOF SYSTEM FREE VIBRATION

- Equation of motion

$$m_1\ddot{x_1} + +(k_1 + k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x_2} - k_2x_1 + (k_2 + k_3)x_2 = 0$$

- Assumed solution

$$x_1(t) = X_1\cos(\omega t + \phi)\,; x_2(t) = X_2\cos(\omega t + \phi)$$
$$\left[\left(-m_1\omega^2 + (k_1 + k_2)\right)X_1 - k_2X_2\right]\cos(\omega t + \phi) = 0$$
$$\left[-k_2X_1 + \left(-m_2\omega^2 + (k_2 + k_3)\right)X_2\right]\cos(\omega t + \phi) = 0$$

TWO DOF SYSTEM FREE VIBRATION

- Matrix form

$$det\begin{bmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{bmatrix} = 0$$

- Solution

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}$$

$$\mp \frac{1}{2}\left[\left\{\frac{(k_1 + k_2)m_2 + (k_2 + k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

TWO DOF SYSTEM FREE VIBRATION

- Magnitude ratio

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2 + (k_2 + k_3)}$$

$$r_2 = \frac{X_2^{(2)}}{X_2^{(2)}} = \frac{-m_1\omega_2^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2 + (k_2 + k_3)}$$

$$\vec{x}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \\ r_1 X_1^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \text{first mode}$$

$$\vec{x}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \\ r_2 X_1^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} = \text{second mode}$$

TWO DOF SYSTEM FREE VIBRATION

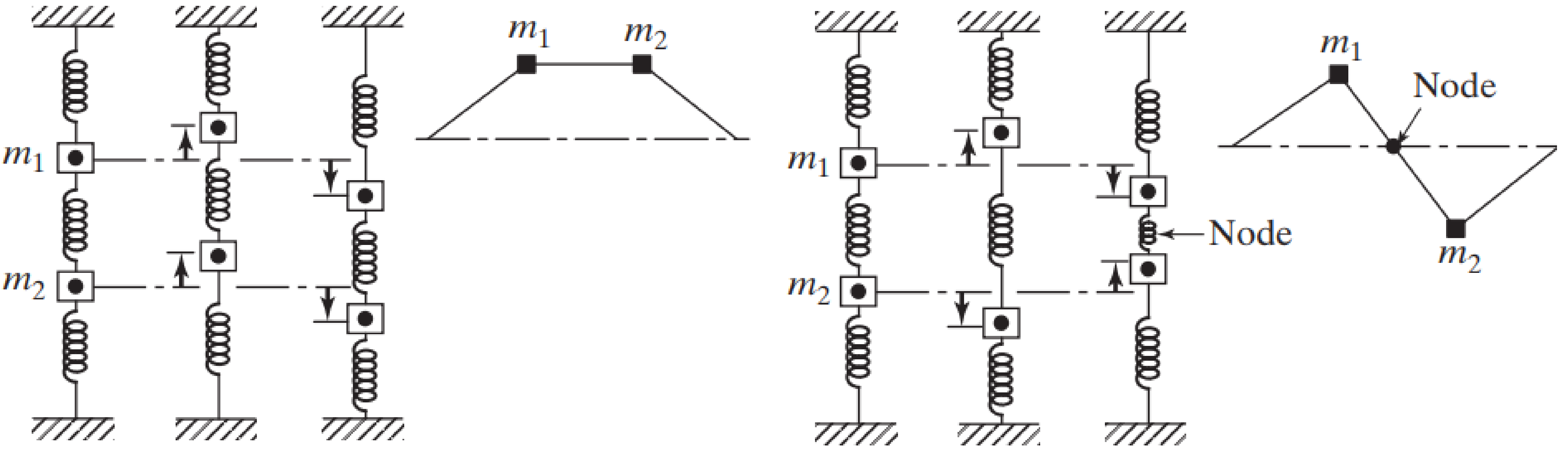
- General Solution

$$\vec{x}^{(1)} = X_1^{(1)} + X_1^{(2)} = X_1^{(1)}\cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2)$$
$$\vec{x}^{(2)} = X_2^{(1)} + X_2^{(2)} = r_1 X_1^{(1)}\cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

- Initial Conditions

$$x_1(0) = X_1^{(1)}\cos(\phi_1) + X_1^{(2)}\cos(\phi_2)$$
$$\dot{x}_1(0) = -\omega_1 X_1^{(1)}\sin(\phi_1) - \omega_2 X_1^{(2)}\sin(\phi_2)$$
$$x_2(0) = r_1 X_1^{(1)}\cos(\phi_1) + r_2 X_1^{(2)}\cos(\phi_2)$$
$$\dot{x}_2(0) = -\omega_1 r_1 X_1^{(1)}\sin(\phi_1) - \omega_2 r_2 X_1^{(2)}\sin(\phi_2)$$

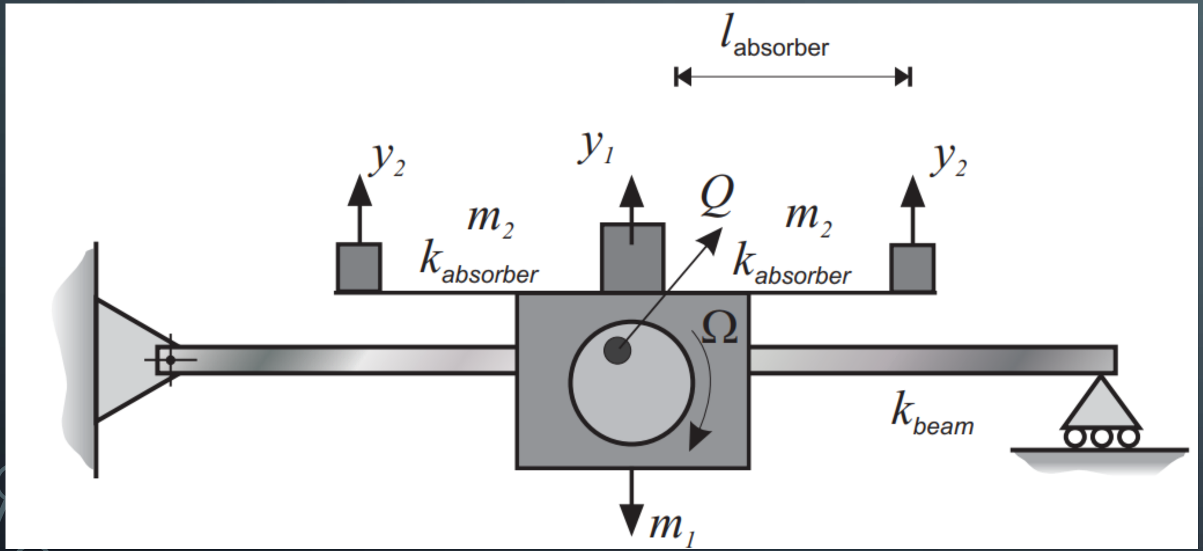
TWO DOF SYSTEM FREE VIBRATION

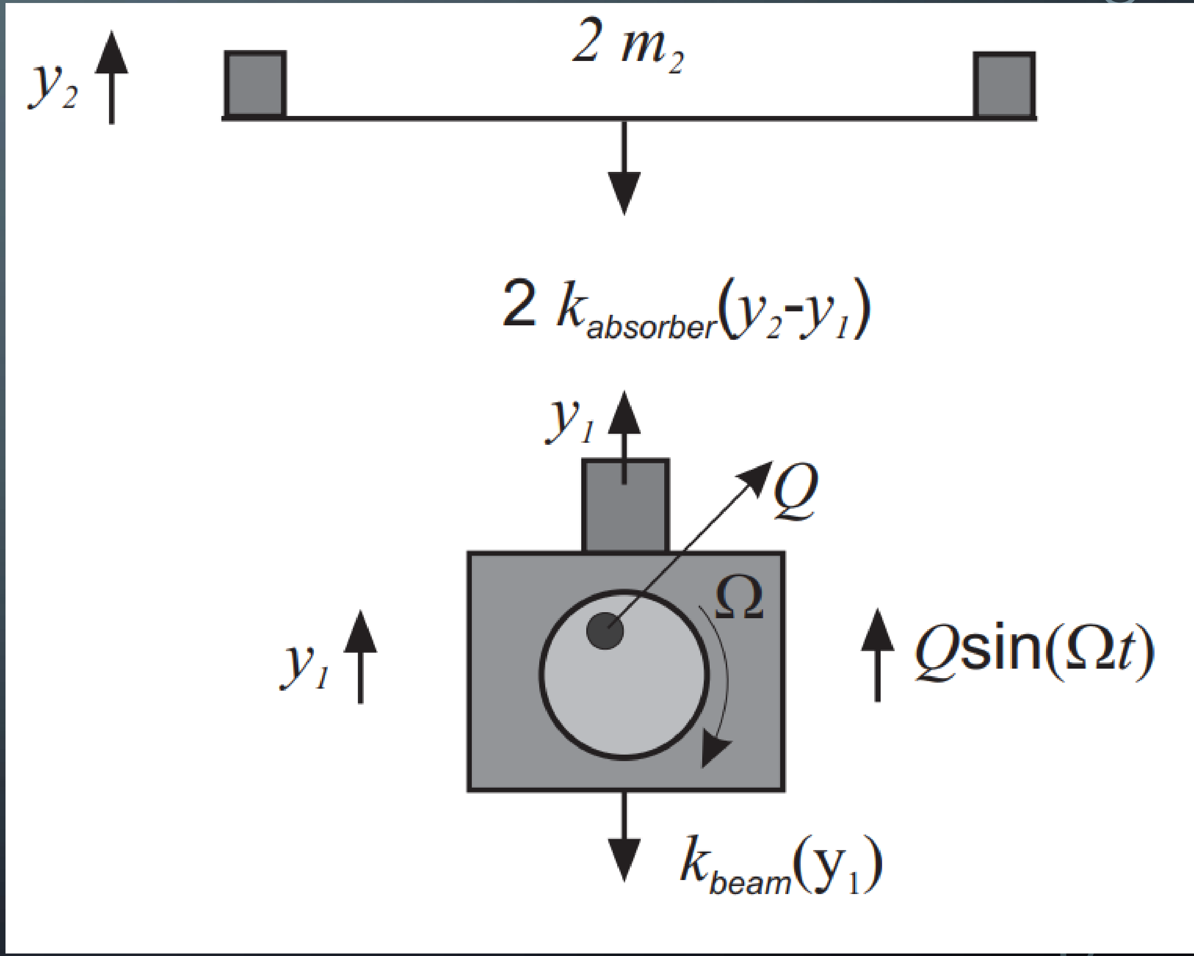


(a) First mode

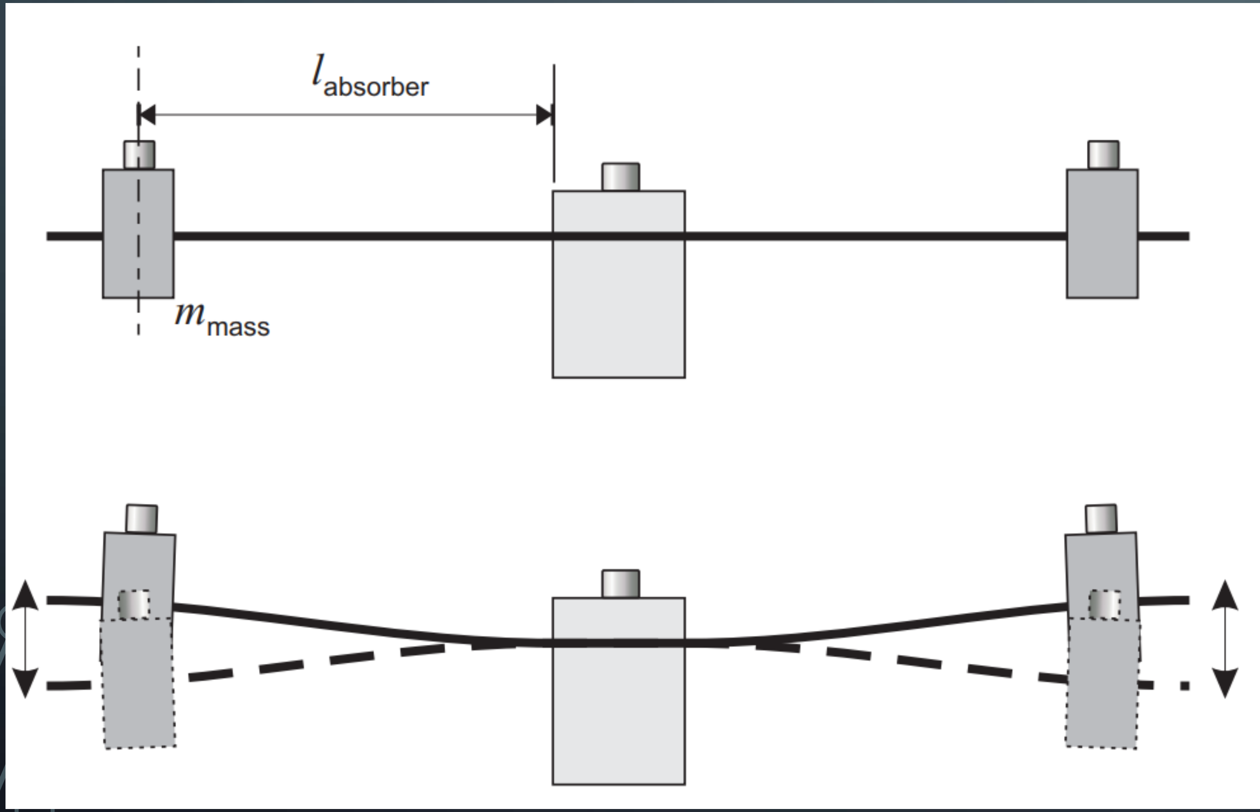
(b) Second mode

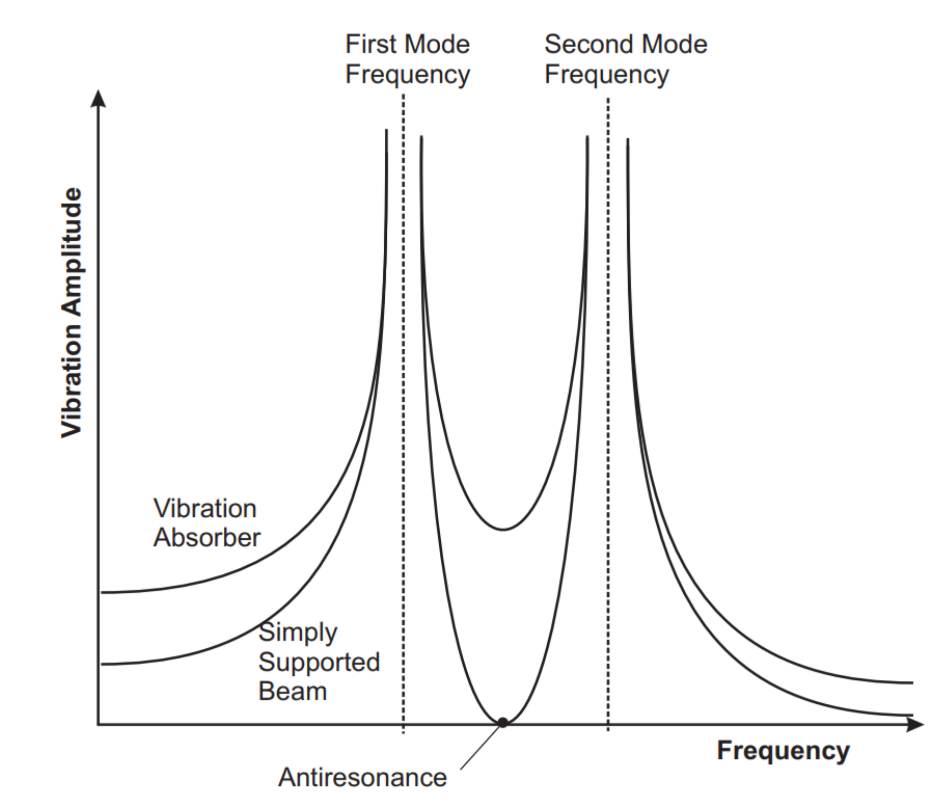
UNDAMPED VIBRATION ABSORBER

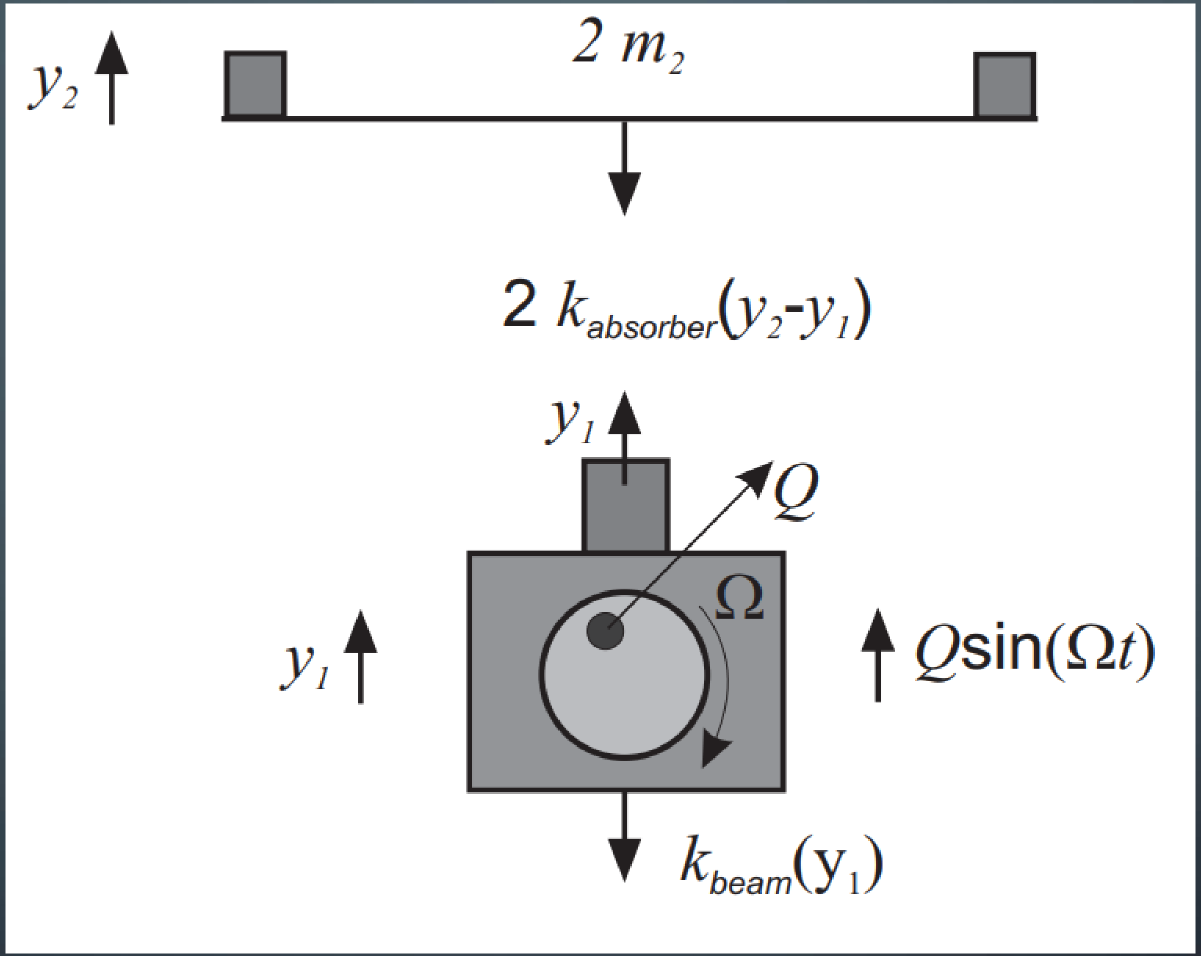




TWO DOF SYSTEM-VIBRATION ABSORBER







$$m_1\ddot{y}_1 + (k_{beam} + 2k_{absorber})y_1 - 2k_{absorber}y_2 = Q\sin(\Omega t)$$
$$2m_2\ddot{y}_2 - 2k_{absorber}y_1 + 2k_{absorber}y_2 = 0$$

TWO DOF SYSTEM-VIBRATION ABSORBER

- Solution

$$y_1 = A\sin\omega t \text{ and } \ddot{y}_1 = -A\omega^2\sin\omega t$$

$$y_2 = B\sin\omega t \text{ and } \ddot{y}_2 = -B\omega^2\sin\omega t$$

$$[(k_{beam} + 2k_{absorber} - m_1\omega^2)A - 2k_{absorber}B]\sin\omega t = Q\sin(\Omega t)$$
$$[-2k_{absorber}A + (2k_{absorber} - 2m_2\omega^2)B]\sin\omega t = 0$$

$$\begin{bmatrix} k_{beam} + 2k_{absorber} - m_1\omega^2 & -2k_{absorber} \\ -2k_{absorber} & 2k_{absorber} - 2m_2\omega^2 \end{bmatrix}\begin{bmatrix} A\sin\omega t \\ B\sin\omega t \end{bmatrix} = \begin{bmatrix} Q\sin(\Omega t) \\ 0 \end{bmatrix}$$
$$CY = F$$

TWO DOF SYSTEM-VIBRATION ABSORBER

- Free response

$$\begin{bmatrix} k_{beam} + 2k_{absorber} - m_1\omega^2 & -2k_{absorber} \\ -2k_{absorber} & 2k_{absorber} - 2m_2\omega^2 \end{bmatrix} = 0$$

$$2m_1m_2\omega^4 - [2m_1k_{absorber} + 2m_2Ck_{beam} + 2k_{absorber}]\omega^2 + 2k_{beam}k_{absorber} = 0$$

- Natural frequency

$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 2m_1m_2; b = -\big(2m_1k_{absorber} + 2m_2(k_{beam} + 2k_{absorber})\big); c = 2k_{beam}k_{absorber}$$

TWO DOF SYSTEM-VIBRATION ABSORBER

$$CY = F$$

$$Y = C^{-1}F = \frac{adj(C)}{\det(C)}$$

$$\begin{bmatrix} A\sin\omega t \\ B\sin\omega t \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} 2k_{absorber} - 2m_2\omega^2 & 2k_{absorber} \\ 2k_{absorber} & k_{beam} + 2k_{absorber} - m_1\omega^2 \end{bmatrix}\begin{bmatrix} Q\sin(\Omega t) \\ 0 \end{bmatrix}$$

$$A = \frac{1}{\Delta}(2k_{absorber} - 2m_2\omega^2)Q$$

$$B = \frac{1}{\Delta}2k_{absorber}Q$$

TWO DOF SYSTEM-VIBRATION ABSORBER

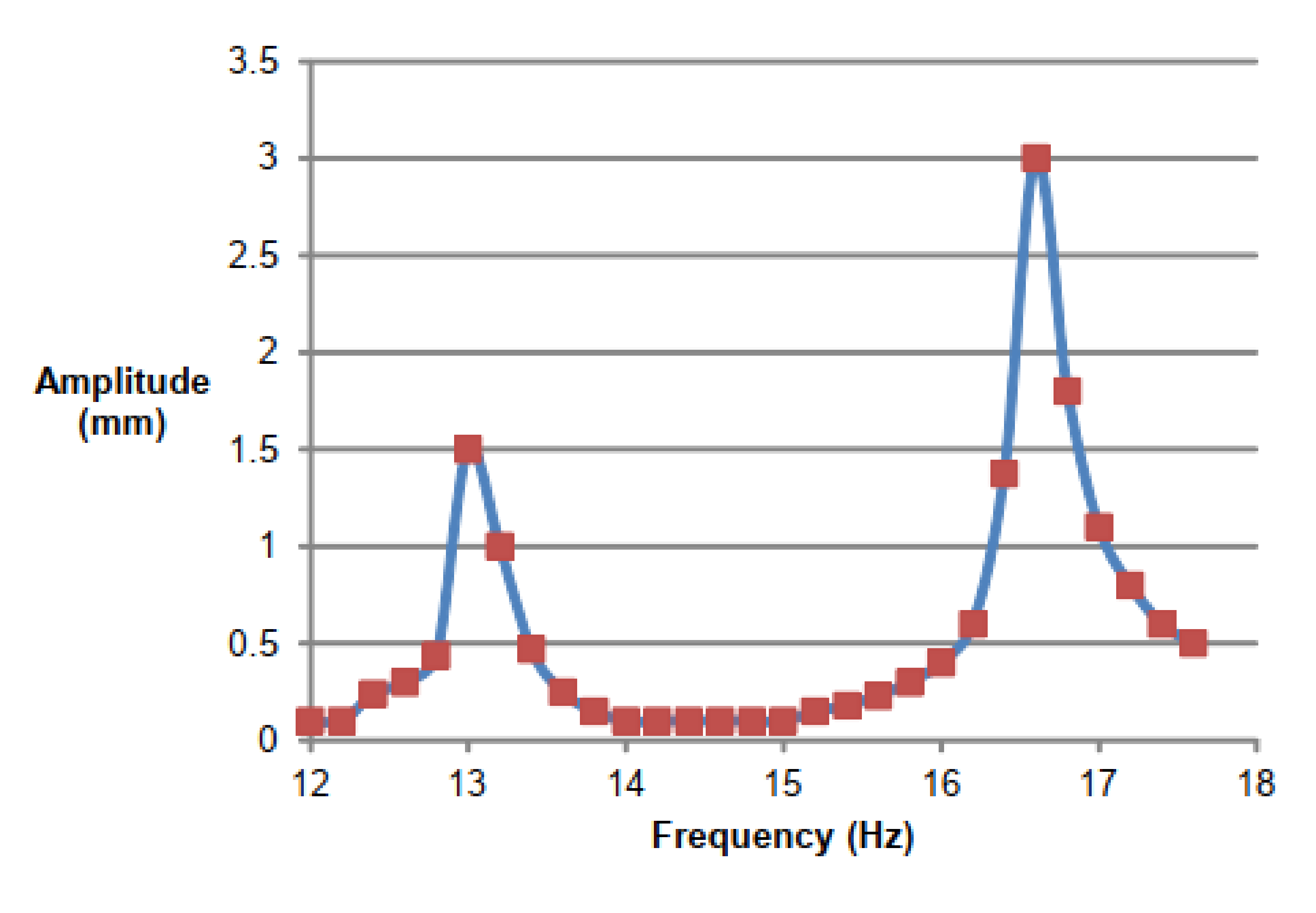
- Objective: reduce amplitude A to zero
- Set A to zero

$$0 = \frac{1}{\Delta}(2k_{absorber} - 2m_2\omega^2)Q$$

- Antiresonance frequency

$$\omega = \sqrt{\frac{k_{absorber}}{m_2}}$$

TWO DOF SYSTEM-VIBRATION ABSORBER

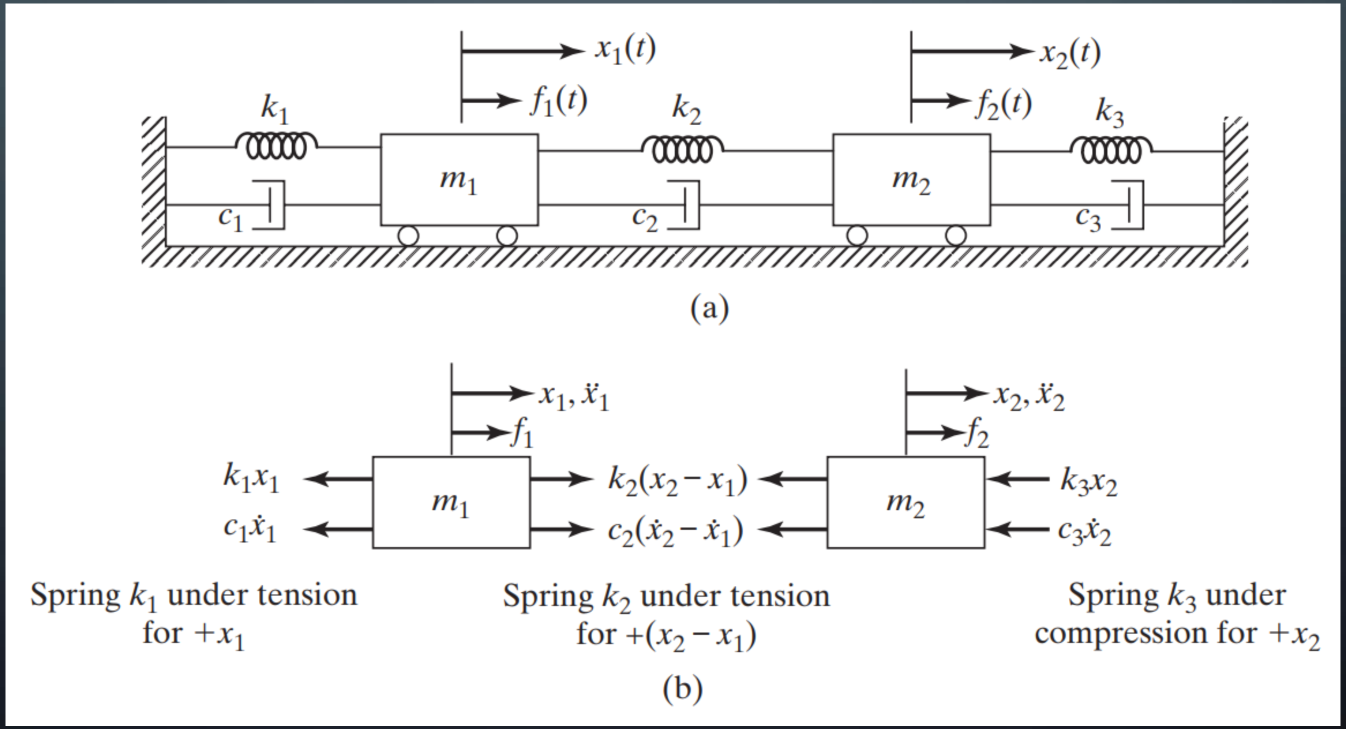


VIDEO DEMO- MODE 1 & MODE 2

VIDEO DEMO-VIBRATION ABSORBER

STUDIO

Find the free-vibration response of the system shown in Figure with $k_1 = 30, k_2 = 5, k_3 = 0,\ m_1 = 10, m_2 =$, and $c_1 = c_2 = c_3 = 0$ for the initial conditions $x_1(0) = 1, \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$



Free-Vibration Response of a Two-Degree-of-Freedom System

Find the free-vibration response of the system shown in Fig. 5.5(a) with $k_1 = 30$, $k_2 = 5$, $k_3 = 0$, $m_1 = 10$, $m_2 = 1$, and $c_1 = c_2 = c_3 = 0$ for the initial conditions $x_1(0) = 1$, $\dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$.

Solution: For the given data, the eigenvalue problem, Eq. (5.8), becomes

$$\begin{bmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

or

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{E.1}$$

By setting the determinant of the coefficient matrix in Eq. (E.1) to zero, we obtain the frequency equation (see Eq. (5.9)):

$$10\omega^4 - 85\omega^2 + 150 = 0 \tag{E.2}$$

from which the natural frequencies can be found as

$$\omega_1^2 = 2.5, \qquad \omega_2^2 = 6.0$$

or

$$\omega_1 = 1.5811, \qquad \omega_2 = 2.4495 \tag{E.3}$$

The substitution of $\omega^2 = \omega_1^2 = 2.5$ in Eq. (E.1) leads to $X_2^{(1)} = 2X_1^{(1)}$, while $\omega^2 = \omega_2^2 = 6.0$ in Eq. (E.1) yields $X_2^{(2)} = -5X_1^{(2)}$. Thus the normal modes (or eigenvectors) are given by

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} X_1^{(1)} \tag{E.4}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -5 \end{Bmatrix} X_1^{(2)} \tag{E.5}$$

The free-vibration responses of the masses m_1 and m_2 are given by (see Eq. (5.15)):

$$x_1(t) = X_1^{(1)} \cos(1.5811t + \phi_1) + X_1^{(2)} \cos(2.4495t + \phi_2) \tag{E.6}$$

$$x_2(t) = 2X_1^{(1)} \cos(1.5811t + \phi_1) - 5X_1^{(2)} \cos(2.4495t + \phi_2) \tag{E.7}$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1, and ϕ_2 are constants to be determined from the initial conditions. By using the given initial conditions in Eqs. (E.6) and (E.7), we obtain

$$x_1(t = 0) = 1 = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2 \tag{E.8}$$

$$x_2(t = 0) = 0 = 2X_1^{(1)} \cos \phi_1 - 5X_1^{(2)} \cos \phi_2 \tag{E.9}$$

$$\dot{x}_1(t = 0) = 0 = -1.5811X_1^{(1)} \sin \phi_1 - 2.4495X_1^{(2)} \sin \phi_2 \tag{E.10}$$

$$\dot{x}_2(t = 0) = -3.1622X_1^{(1)} \sin \phi_1 + 12.2475X_1^{(2)} \sin \phi_2 \tag{E.11}$$

The solution of Eqs. (E.8) and (E.9) yields

$$X_1^{(1)} \cos \phi_1 = \frac{5}{7}, \qquad X_1^{(2)} \cos \phi_2 = \frac{2}{7} \tag{E.12}$$

$$x_1(t = 0) = 1 = X_1^{(1)} \cos \phi_1 + X_1^{(2)} \cos \phi_2 \tag{E.8}$$

$$x_2(t = 0) = 0 = 2X_1^{(1)} \cos \phi_1 - 5X_1^{(2)} \cos \phi_2 \tag{E.9}$$

$$\dot{x}_1(t = 0) = 0 = -1.5811X_1^{(1)} \sin \phi_1 - 2.4495X_1^{(2)} \sin \phi_2 \tag{E.10}$$

$$\dot{x}_2(t = 0) = -3.1622X_1^{(1)} \sin \phi_1 + 12.2475X_1^{(2)} \sin \phi_2 \tag{E.11}$$

The solution of Eqs. (E.8) and (E.9) yields

$$X_1^{(1)} \cos \phi_1 = \frac{5}{7}, \qquad X_1^{(2)} \cos \phi_2 = \frac{2}{7} \tag{E.12}$$

while the solution of Eqs. (E.10) and (E.11) leads to

$$X_1^{(1)} \sin \phi_1 = 0, \qquad X_1^{(2)} \sin \phi_2 = 0 \tag{E.13}$$

Equations (E.12) and (E.13) give

$$X_1^{(1)} = \frac{5}{7}, \qquad X_1^{(2)} = \frac{2}{7}, \qquad \phi_1 = 0, \qquad \phi_2 = 0 \tag{E.14}$$

Thus the free-vibration responses of m_1 and m_2 are given by

$$x_1(t) = \frac{5}{7} \cos 1.5811t + \frac{2}{7} \cos 2.4495t \tag{E.15}$$

$$x_2(t) = \frac{10}{7} \cos 1.5811t - \frac{10}{7} \cos 2.4495t \tag{E.16}$$

The graphical representation of Eqs. (E.15) and (E.16) is considered in Example 5.17.