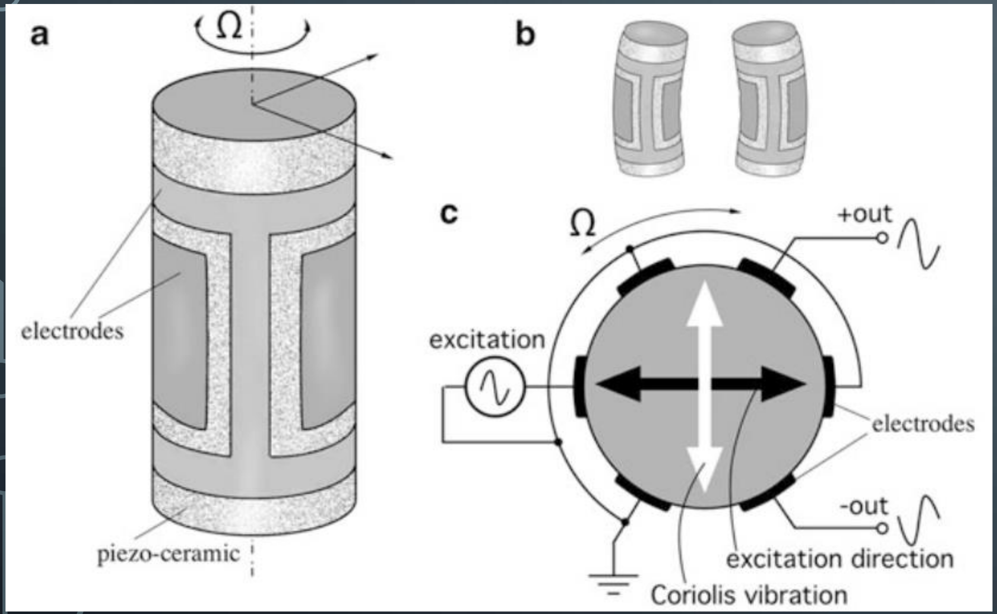
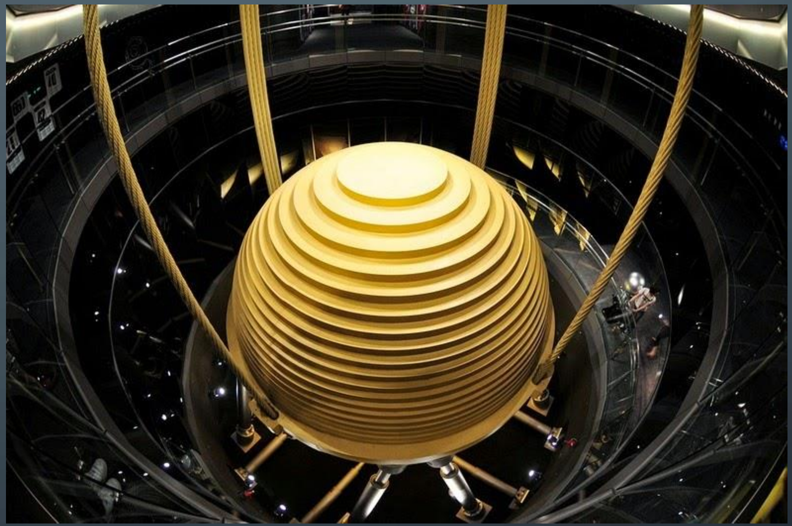
FORCED AND FREE VIBRATIONS

LAB 1

APPLICATIONS



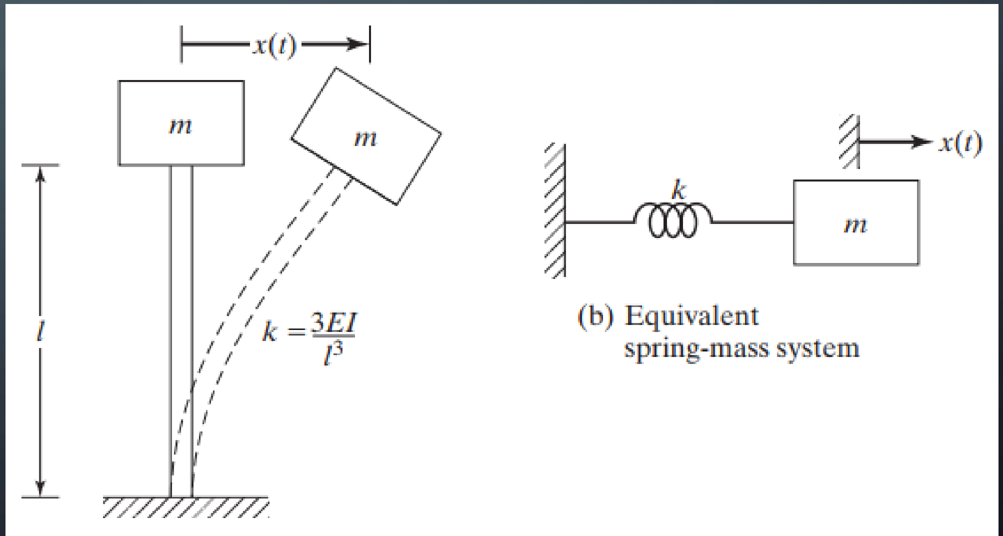


FORCED AND FREE VIBRATIONS

- Free Vibration
- Vibration Analysis Procedure
- Free Vibrations SDOF
- Beam and Spring System
- Damping
- Forced Vibrations
  - Single Degree of Freedom
  - Beam and Spring System

FREE VIBRATION

- A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting after the initial disturbance.

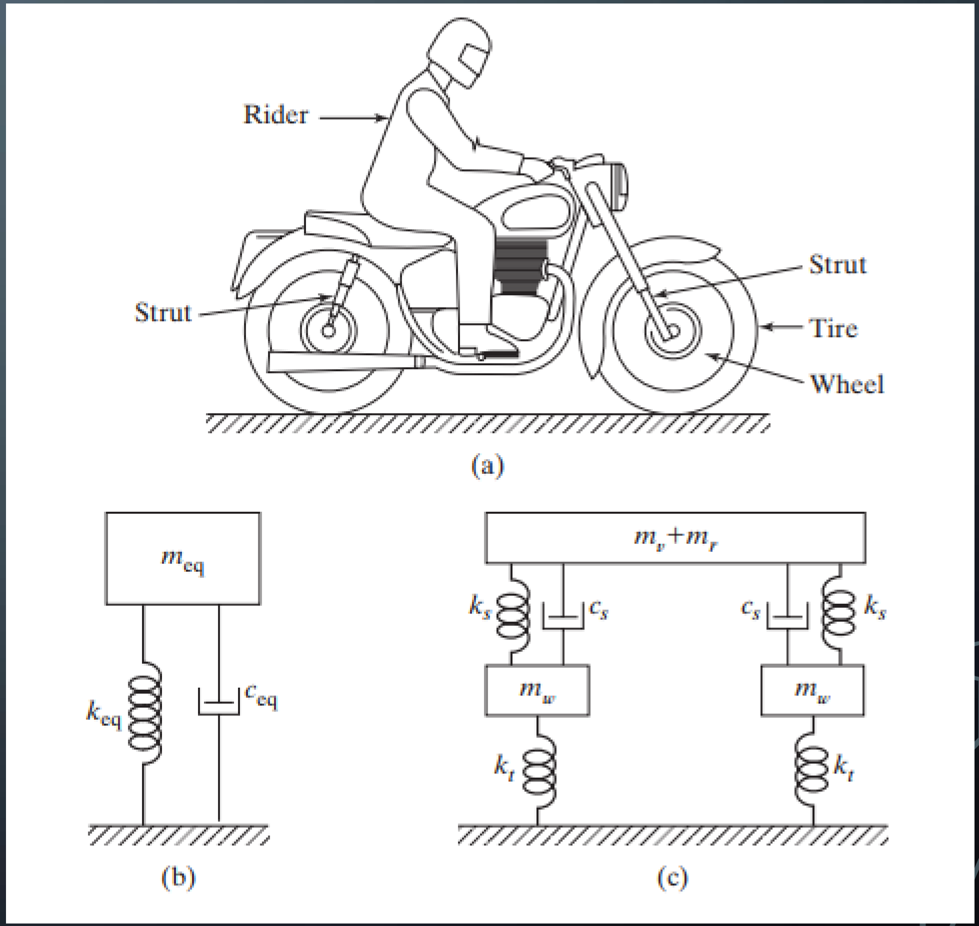


(a) Idealization of the tall structure

(b) Equivalent spring-mass system

VIBRATION ANALYSIS PROCEDURE

- Step 1: Mathematical Modeling
- Step 2: Derivation of Governing Equations
- Step 3: Solution of the Governing Equations
- Step 4: Interpretation of the Results

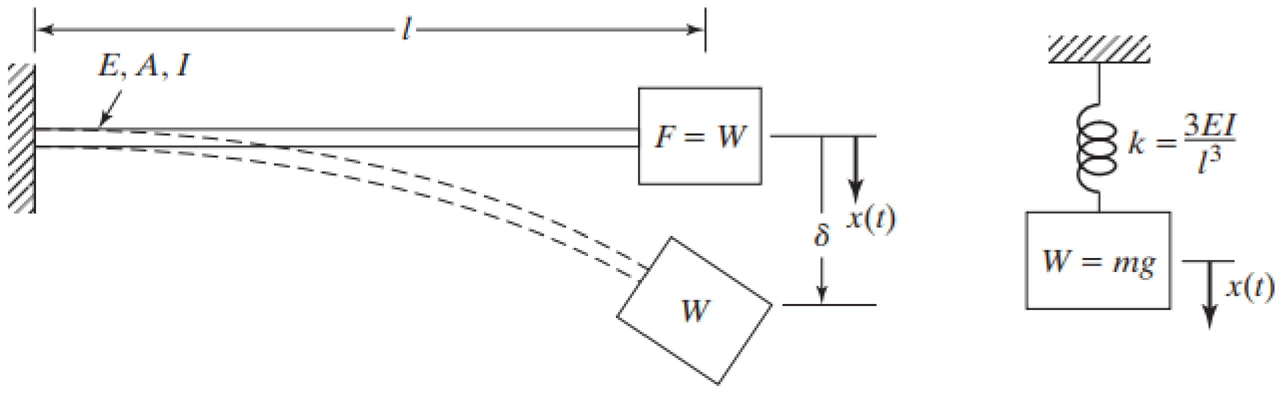


FREE VIBRATIONS SDOF

- Cantilever beam end deflection

$$\delta = \frac{Wl^3}{3EI}$$

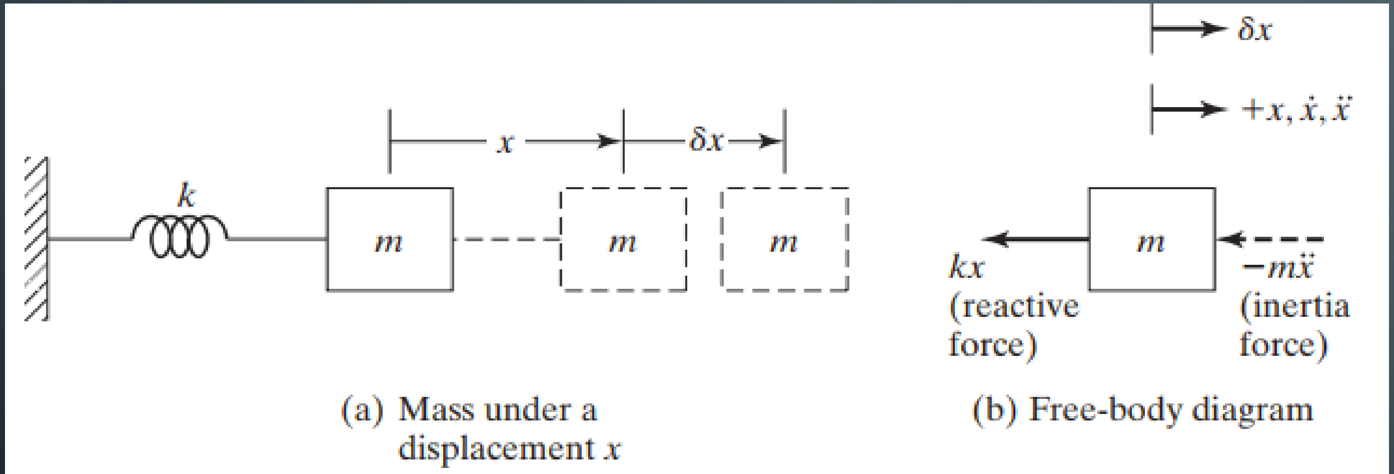
$$k = \frac{W}{\delta} = \frac{3EI}{l^3}$$



(a) Cantilever with end force

(b) Equivalent spring

FREE VIBRATIONS SDOF



- Step 2

$$\Sigma F_x = m\ddot{x} = -kx$$
$$m\ddot{x} + kx = 0$$

- Step 3

$$x(t) = Ce^{st}$$
$$C(ms^2 + k) = 0$$
$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n$$
$$\omega_n = \sqrt{\frac{k}{m}}$$
$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$$

- Euler's Formula

$$e^{i\alpha t} = \cos \alpha t + \pm i \sin \alpha t$$
$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

FREE VIBRATIONS SDOF

- Initial conditions

$$x(t = 0) = x_0 = A_1$$
$$\dot{x}(t = 0) = \dot{x}_0 = \omega_n A_2$$

- Solution

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

- Change of variable

$$A_1 \; = \; A \cos \phi \, ; A_2 \; = \; A \sin \phi$$

- Amplitude & Phase

$$A = \sqrt{A_1^2 + A_2^2}$$

$$\phi = \tan^{-1} (\frac{A_2}{A_1})$$

- Solution

$$x(t) = A \cos(\omega_n t - \phi)$$

LAB: FREE VIBRATION OF A BEAM AND SPRING

- Equation of motion

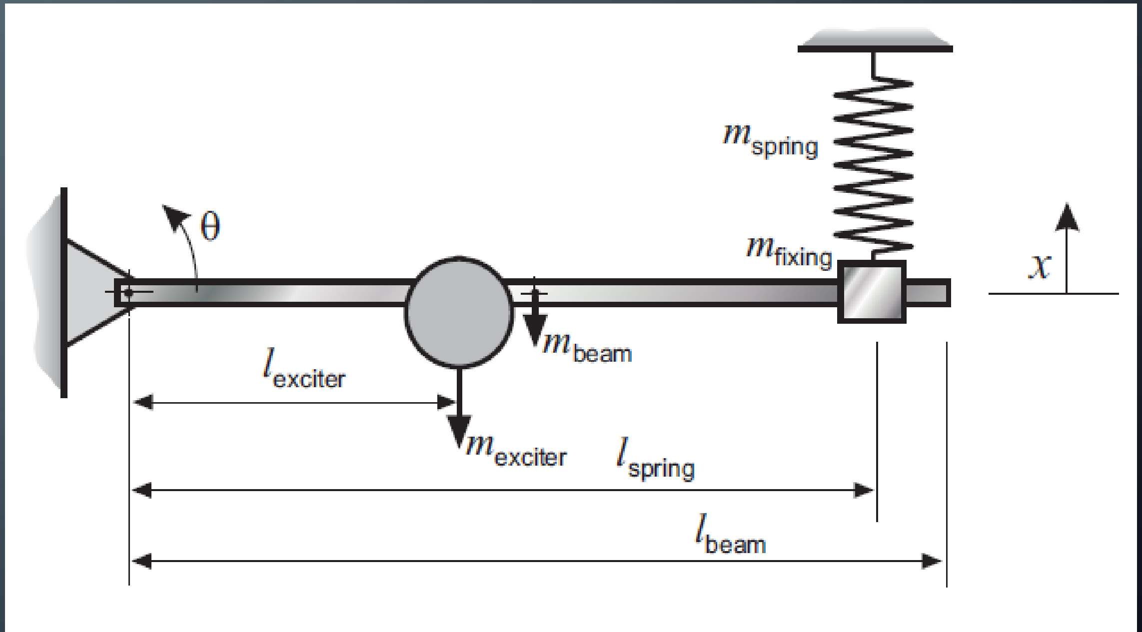
$$I_A\ddot{\theta} + kx_s l_{spring} = 0$$

- Small angle approximation

$$x_s = \theta l_{spring}$$

- Total mass moment of inertia

$$I_A = I_{beam} + I_{spring} + I_{exciter}$$



$$I_A = \frac{1}{3}m_{beam}l_{beam}^2 + \left(\frac{m_{spring}}{3} + m_{fixing}\right)l_{spring}^2 + m_{exciter}l_{exciter}^2$$

LAB: FREE VIBRATION OF A BEAM AND SPRING

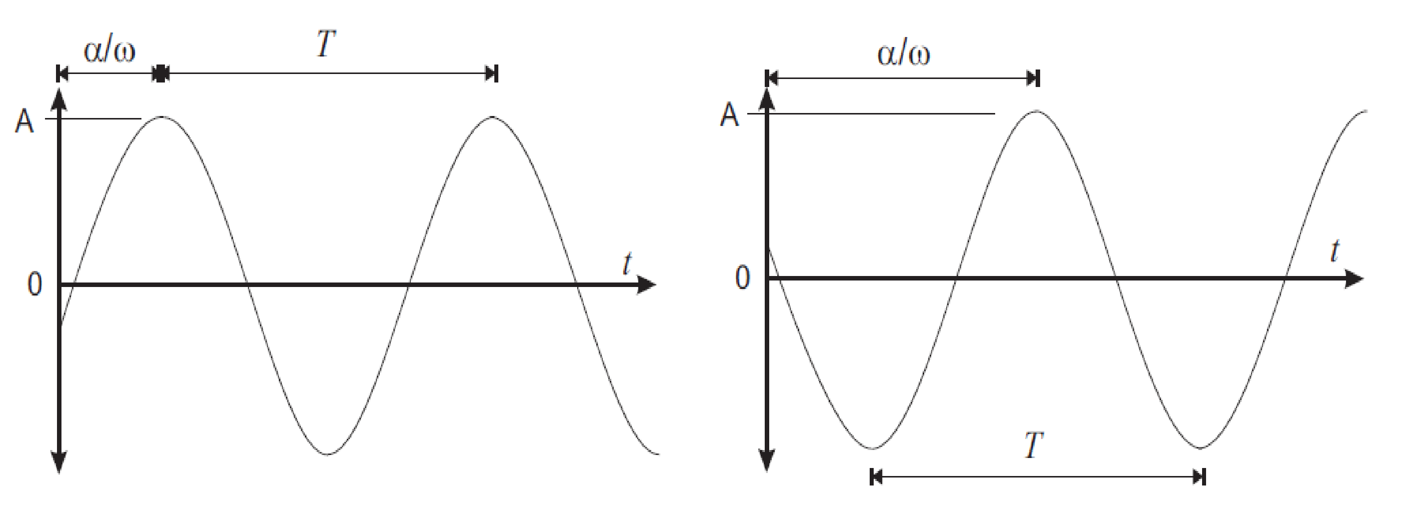
- Solution

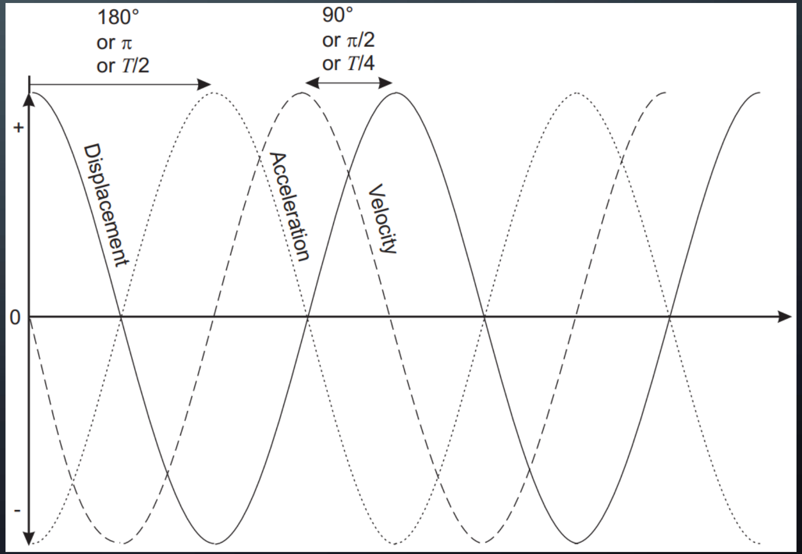
$$\theta = \mathrm{A}\cos\left(\sqrt{\frac{kl_{spring}^2}{I_A}}t - \alpha\right)$$

- Period & Natural frequency

$$\mathrm{T} = 2\pi\sqrt{\frac{I_A}{kl_{spring}^2}}$$

$$\omega = \sqrt{\frac{kl_{spring}^2}{I_A}}$$





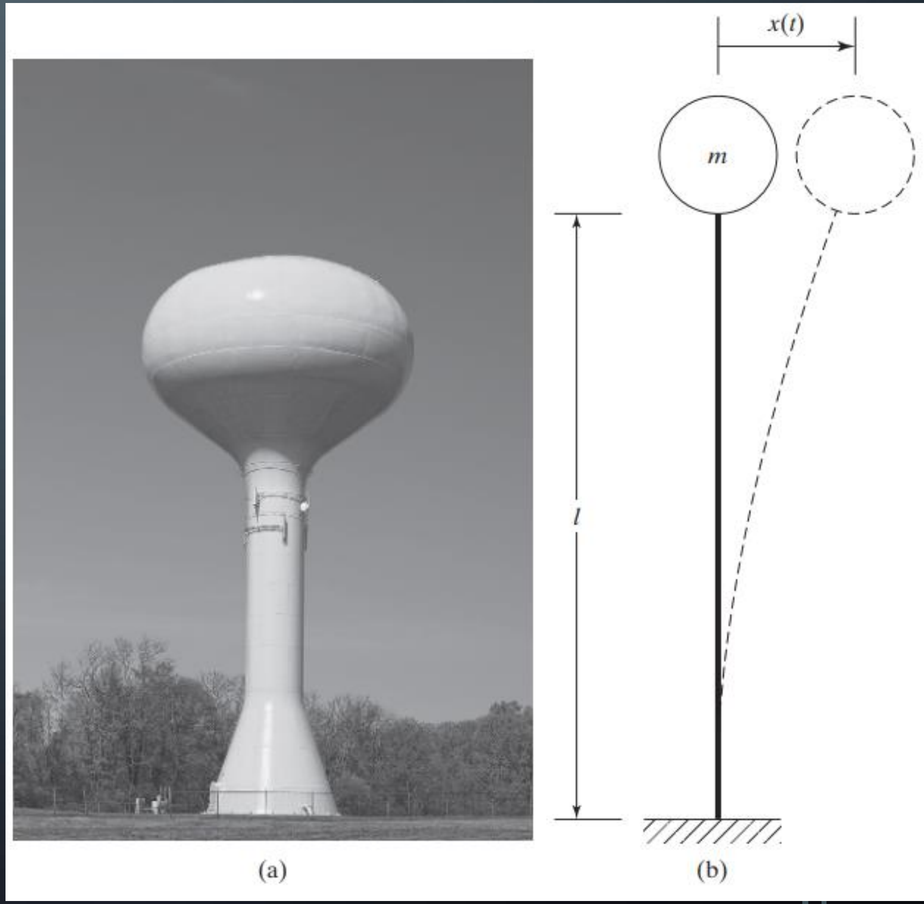
STUDIO

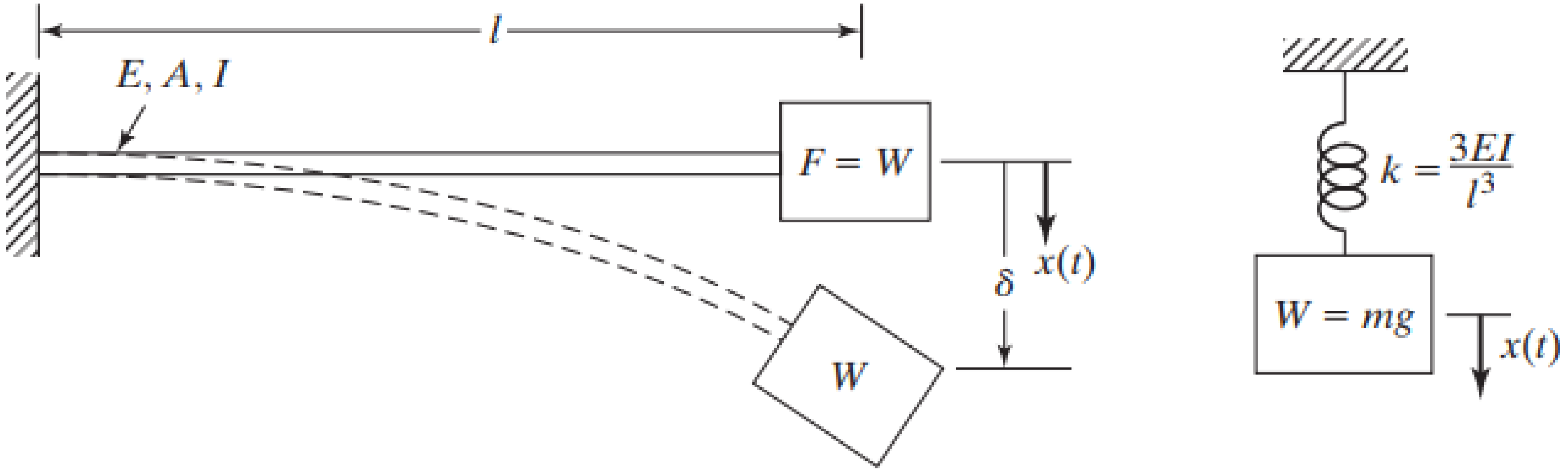
The column of the water tank shown in Figure is 100 m high and is made of reinforced concrete with a tubular cross section of inner diameter 2.5 m and outer diameter 3 m. The tank has a mass of 275,000 kg when filled with water. By neglecting the mass of the column and assuming the Young's modulus of reinforced concrete as 30 GPa, determine the following:

a. The natural frequency and the natural time period of transverse vibration of the water tank.

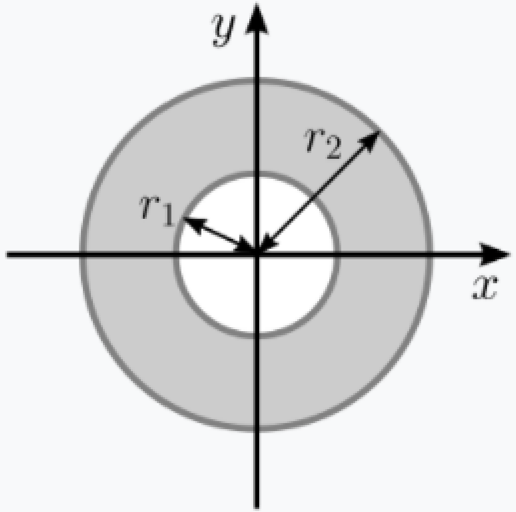
b. The vibration response of the water tank due to an initial transverse displacement of 25 cm.

c. The maximum values of the velocity and acceleration experienced by the water tank

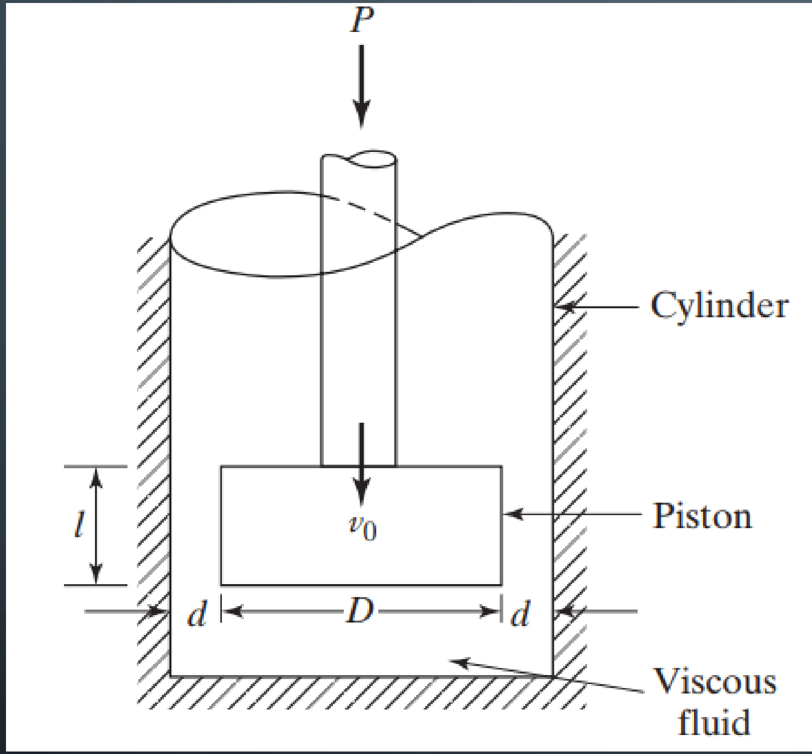


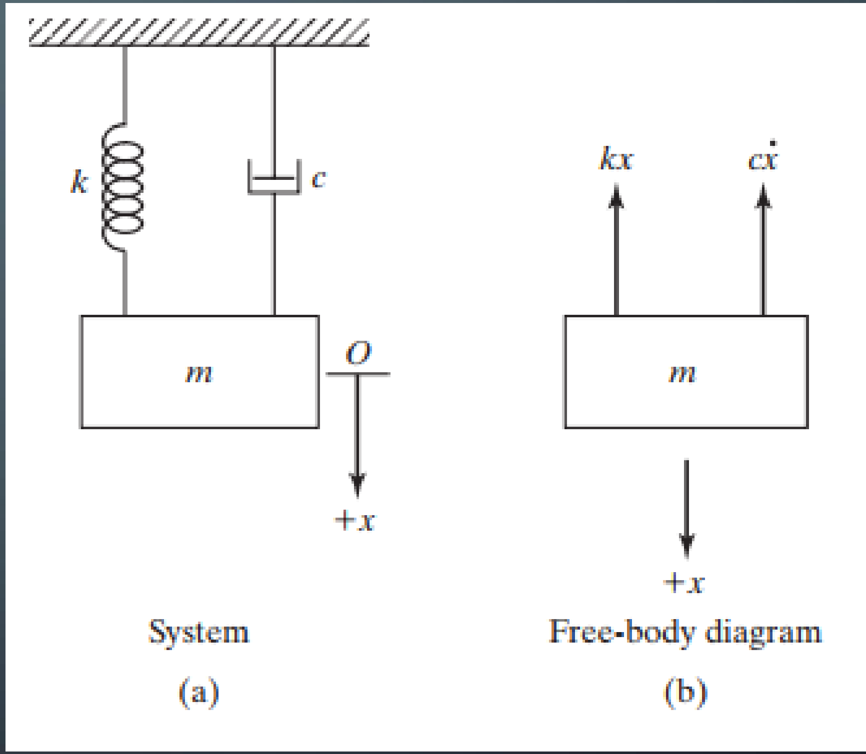


(a) Cantilever with end force (b) Equivalent spring

An annulus of inner radius r_1 and outer radius r_2		$I_x = \frac{\pi}{4}\left({r_2}^4 - {r_1}^4\right)$ $I_y = \frac{\pi}{4}\left({r_2}^4 - {r_1}^4\right)$ $I_z = \frac{\pi}{2}\left({r_2}^4 - {r_1}^4\right)$	For thin tubes, $r \equiv r_1 \approx r_2$ and $r_2 \equiv r_1 + t$. So, for a thin tube, $I_x = I_y \approx \pi r^3 t$. I_z is the Polar moment of inertia.

FREE VIBRATION WITH DAMPING





$$F = -c\dot{x}$$
$$m\ddot{x} + c\dot{x} + kx = 0$$

FREE VIBRATION WITH DAMPING

- Assume a solution in the form

$$x(t) = Ce^{st}$$

- Characteristic equation

$$ms^2 + cs + k = 0$$

- Roots & damping ratio

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n$$

$$\zeta = c/c_c$$

FREE VIBRATION WITH DAMPING

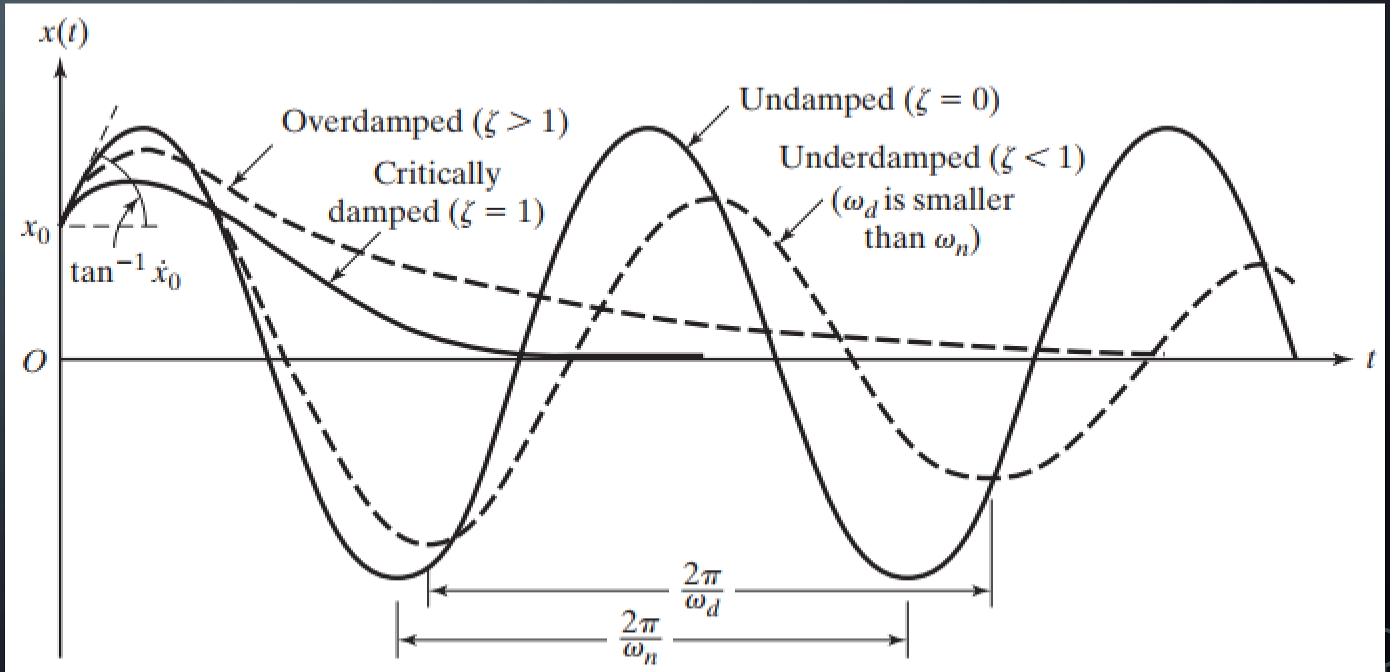
$$y(t) = C_1 e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t - \phi\right) \qquad 0 \le \zeta < 1$$

$$y(t) = (C_1 + C_2 t)e^{-\omega_n t} \qquad \zeta = 1$$

$$y(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \qquad \zeta > 1$$

The response oscillation varies with different damping ratio

Damped natural frequency (Ringing frequency): $\omega_d = \omega_n\sqrt{1-\zeta^2}$

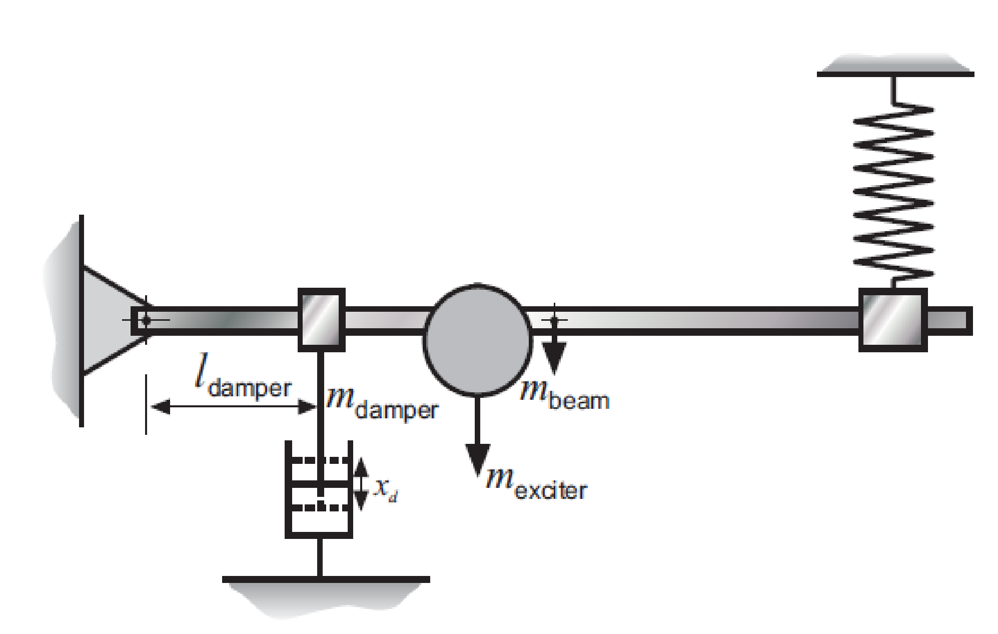


LAB: FREE VIBRATION OF A BEAM AND SPRING WITH DAMPING

$$I_A\ddot{\theta} = -kx_s l_{spring} - cl_{damper}\dot{x}_d$$

$$\ddot{\theta} + \frac{cl_{damper}^2}{I_A}\dot{\theta} + \frac{kl_{spring}^2}{I_A}\theta = 0$$

$$I_A = I_{beam} + I_{spring} + I_{exciter} + I_{damper}$$



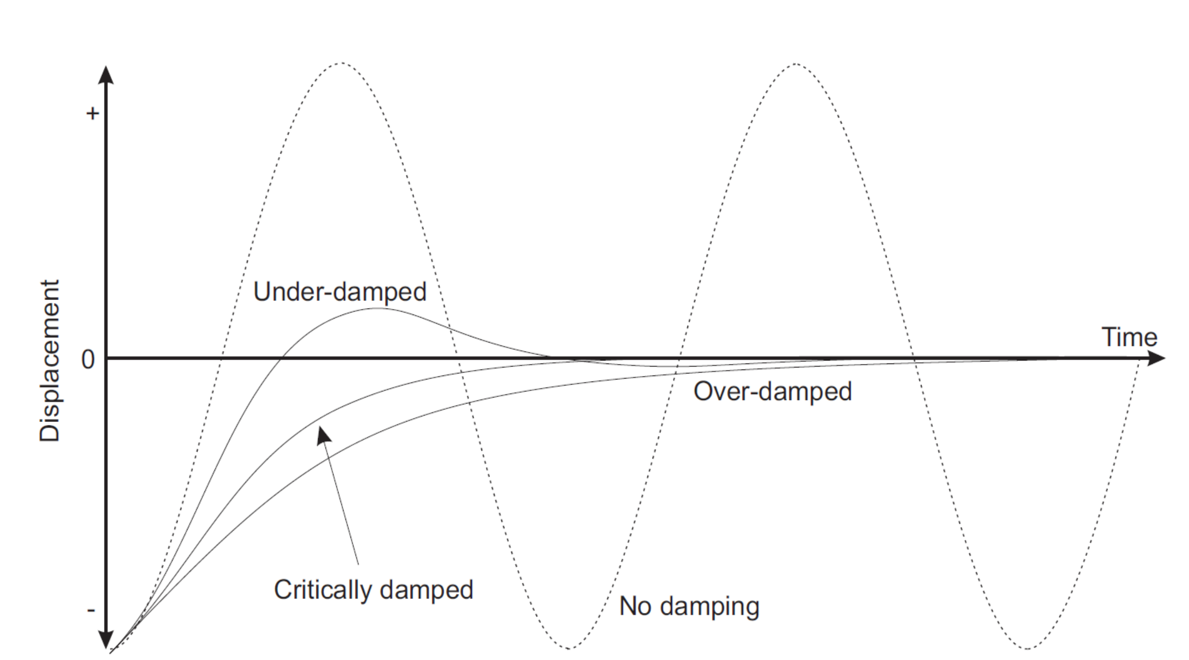
$$\theta = Ce^{rt}$$

$$r^2 + 2\gamma r + \omega^2 = 0$$

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

γ:decay coefficient

LAB: FREE VIBRATION OF A BEAM AND SPRING WITH DAMPING

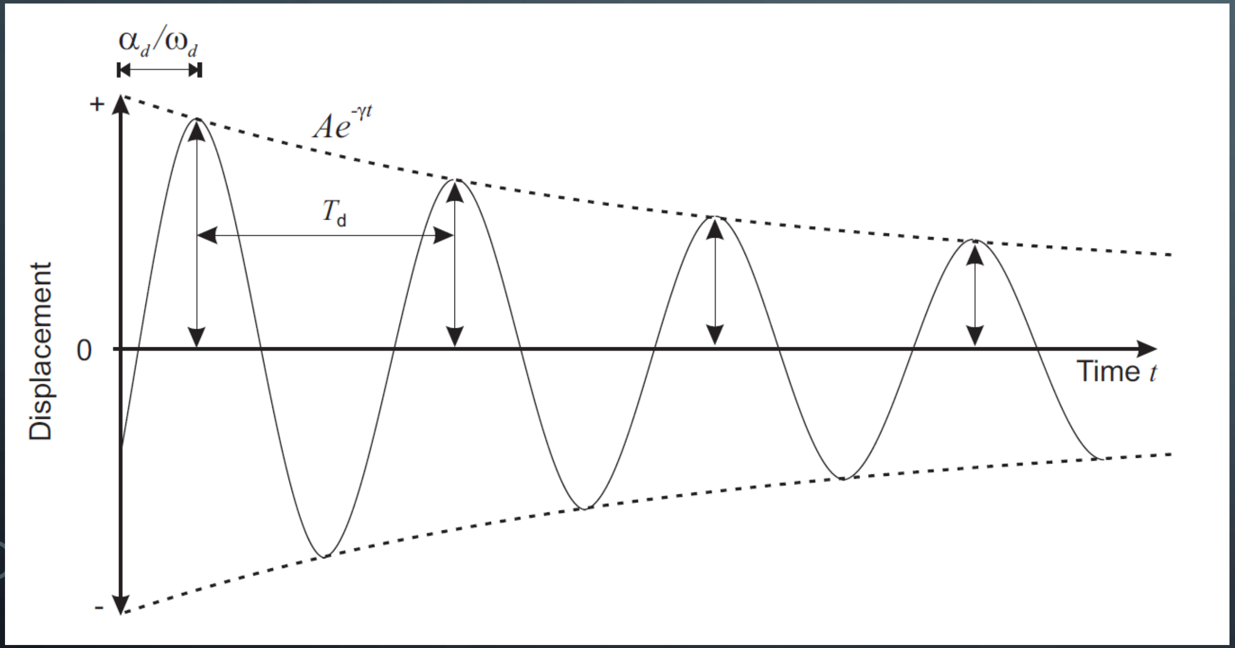


Damping Condition	Damping coefficient	Damping Ratio	Displacement Equation
Undamped	$c = 0$	$\zeta = 0$	$\theta = A\cos(\omega t - \alpha)$
Underdamped	$c < c_c$	$0 < \zeta < 1$	$\theta = Ae^{-\gamma t}\cos\left(\sqrt{\omega^2 - \gamma^2}t - \alpha_d\right)$
Critically Damped	$c = c_c$	$\zeta = 1$	$\theta = (C_1 + C_2 t)e^{-\gamma t}$
Overdamped	$c > c_c$	$\zeta > 1$	$\theta = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

LAB: FREE VIBRATION OF A BEAM AND SPRING WITH DAMPING

- Underdamped, where $\gamma < \omega$, oscillates with a gradually reducing amplitude until equilibrium is reached.
- Critically damped, where $\gamma = \omega$, once displaced the system will return to equilibrium in the shortest possible time without oscillating
- Overdamped, where $\gamma > \omega$, Oscillation will not occur and a gradual return to equilibrium will occur at a rate slower than the critically damped situation.

LAB: FREE VIBRATION OF A BEAM AND SPRING WITH DAMPING

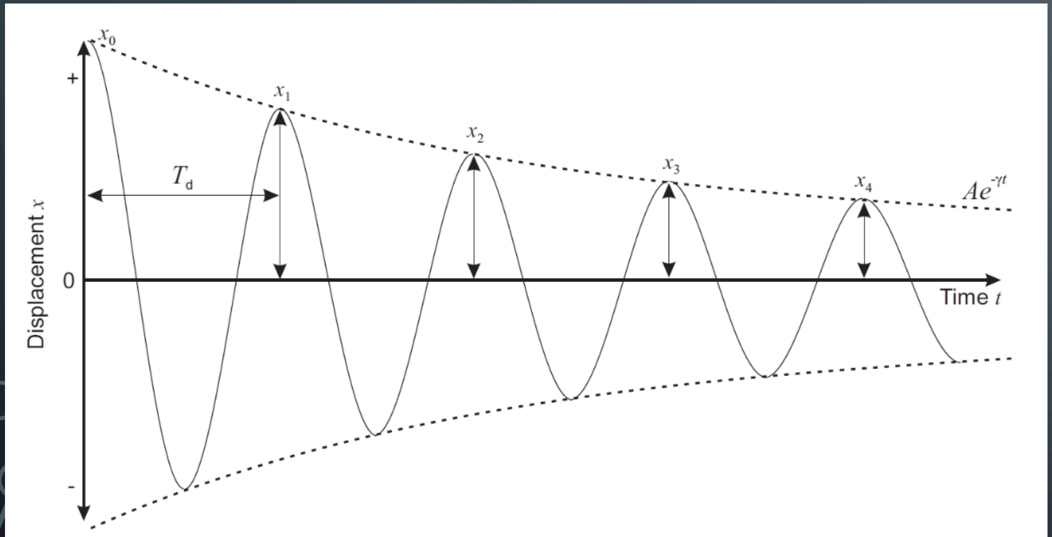


$$\theta = Ae^{-\gamma t}\cos\left(\sqrt{\omega^2 - \gamma^2}t - \alpha_d\right)$$

$$\omega_d^2 = \sqrt{\omega^2 - \gamma^2}$$

$$\frac{\omega_d}{\omega} = \sqrt{1 - \zeta^2}$$

LOGARITHMIC DECREMENT APPROXIMATION

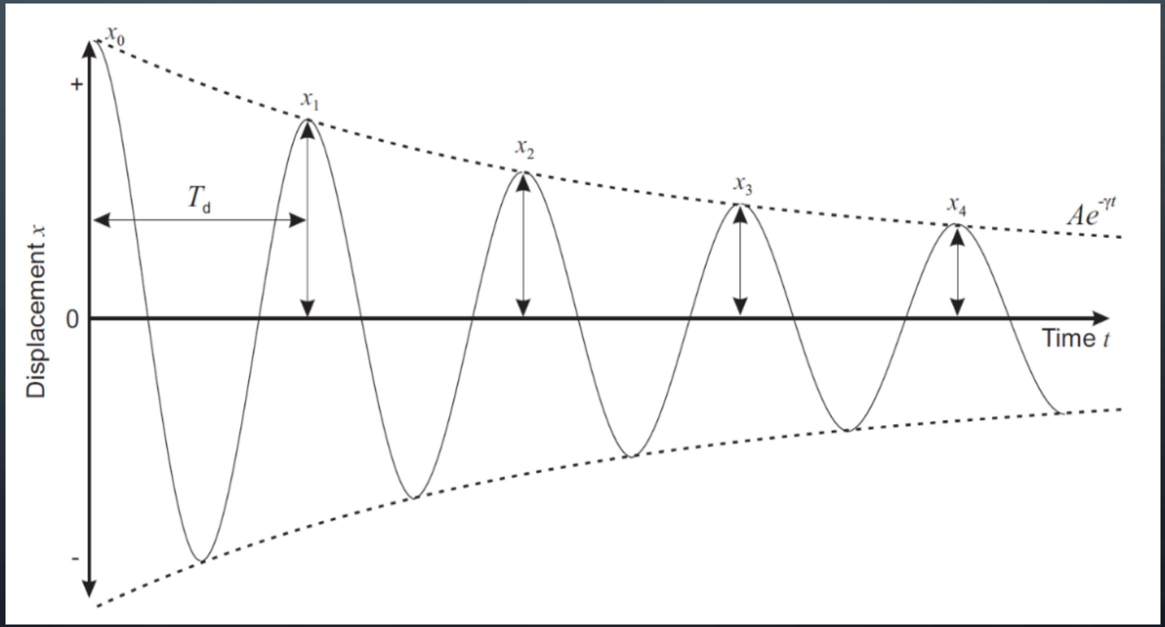


$$\frac{x_0}{x_{0+j}} = \frac{Ae^{-\gamma t_0}}{Ae^{-\gamma(t_0+jT_d)}} = e^{\gamma j T_d} = e^{j\delta}$$

$$\delta = \frac{1}{j}\ln\frac{x_0}{x_{0+j}} = \gamma T_d = \frac{2\pi\gamma}{\omega_d}$$

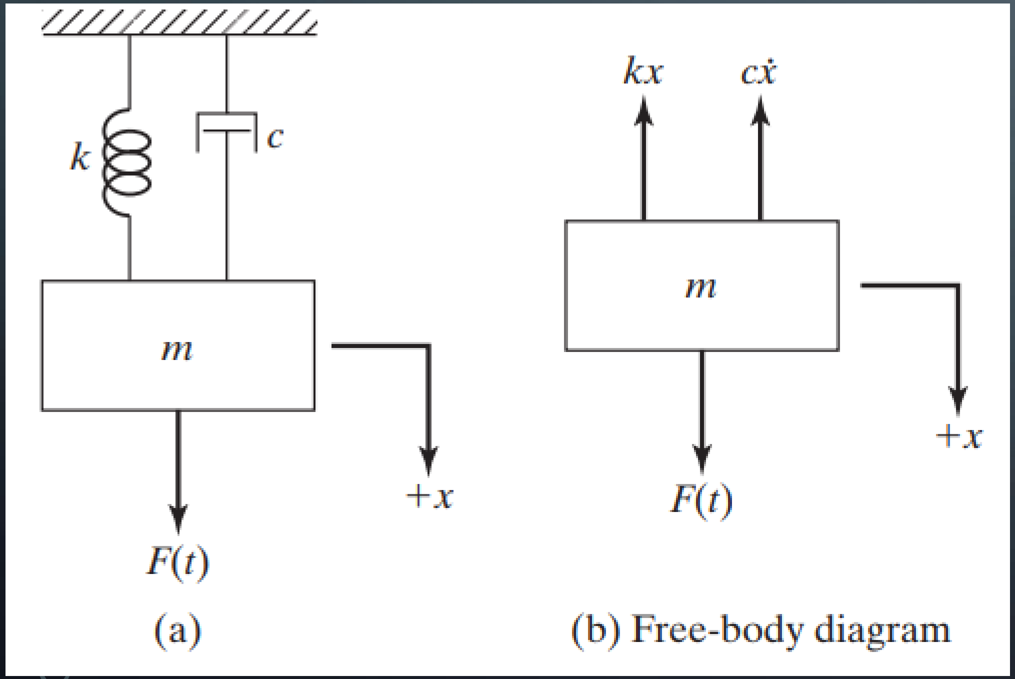
STUDIO

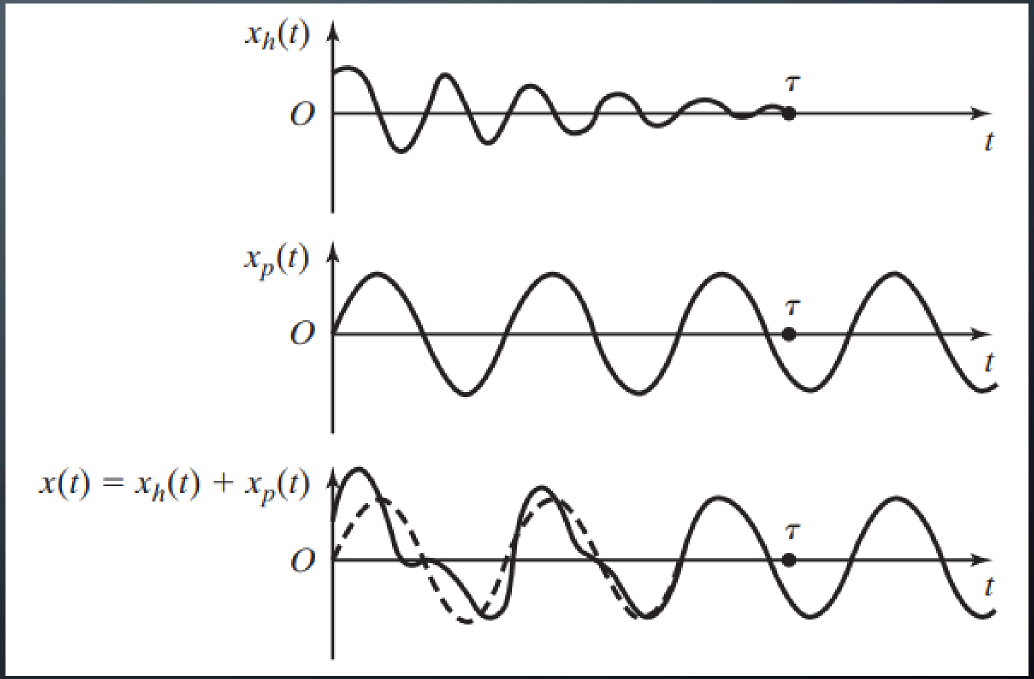
- From the experiments, it is measured that $x_0 = 100$ mm, $x_2 = 8.046$ mm and $T_d = 1.02\ s$. Find the natural frequency of the system in units of Hz.



FORCED VIBRATIONS SDOF

- Forced Vibration. If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as forced vibration.





FORCED VIBRATIONS SDOF

$$m\ddot{x} + kx = F_0 \cos \omega t$$

- Homogeneous and particular solutions

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t$$
$$x_p(t) = X \cos \omega t$$

- Solution

$$X = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t$$

FORCED VIBRATIONS SDOF

- Initial conditions

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}$$
$$C_2 = \dot{x}_0/\omega_n$$

- Solution

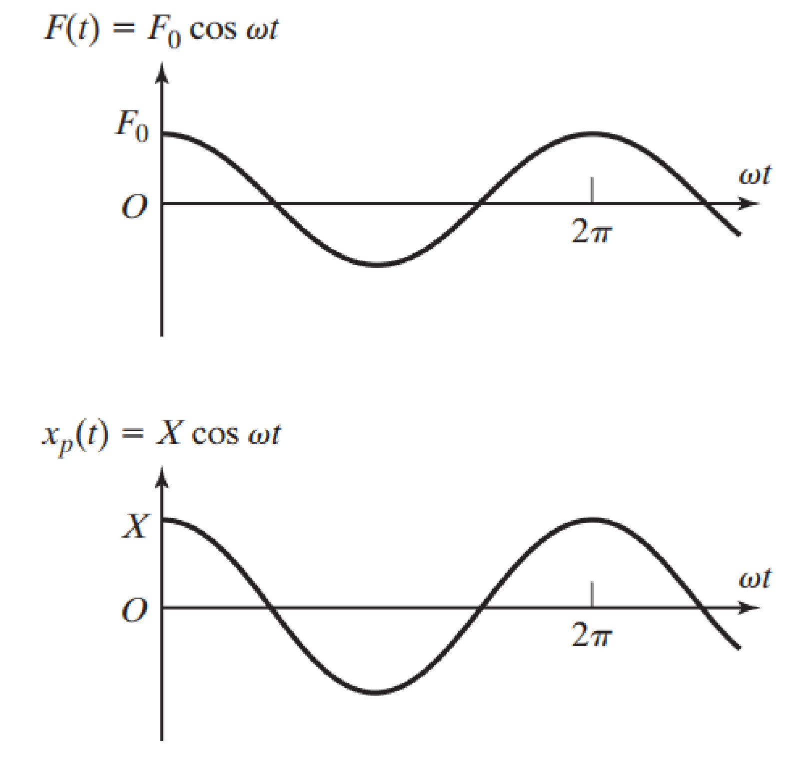
$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n\, t + (\dot{x}_0/\omega_n)\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t$$

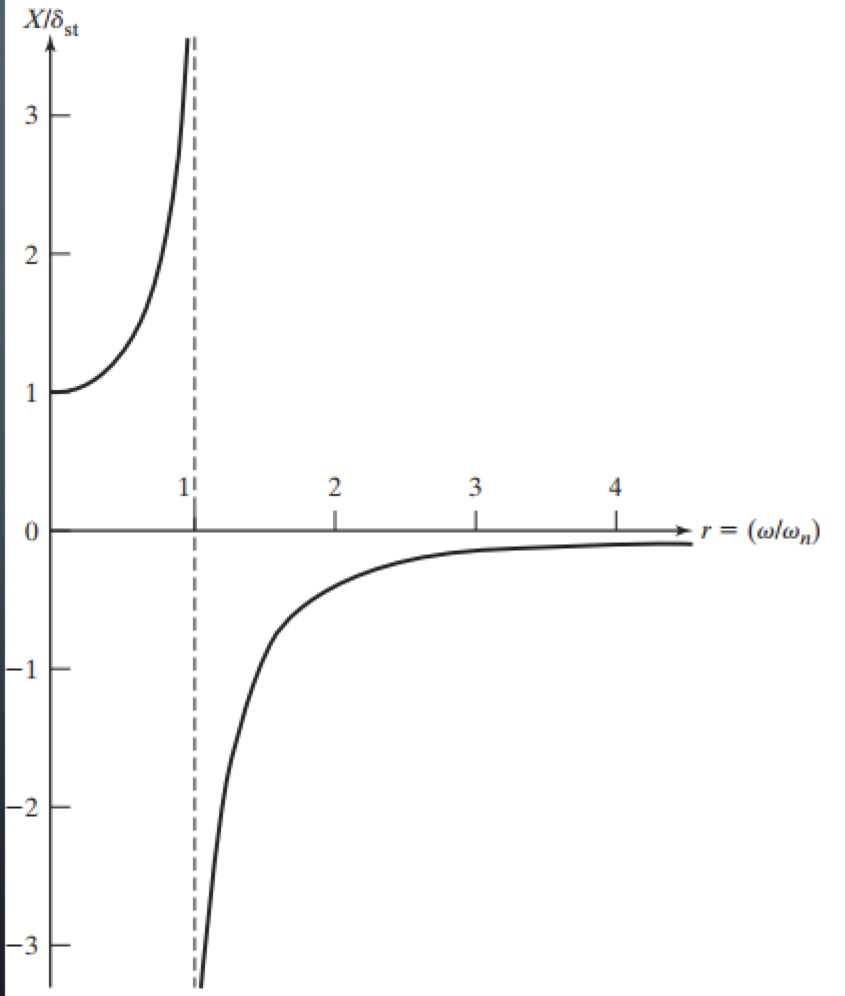
- Amplification factor

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

FORCED VIBRATIONS SDOF

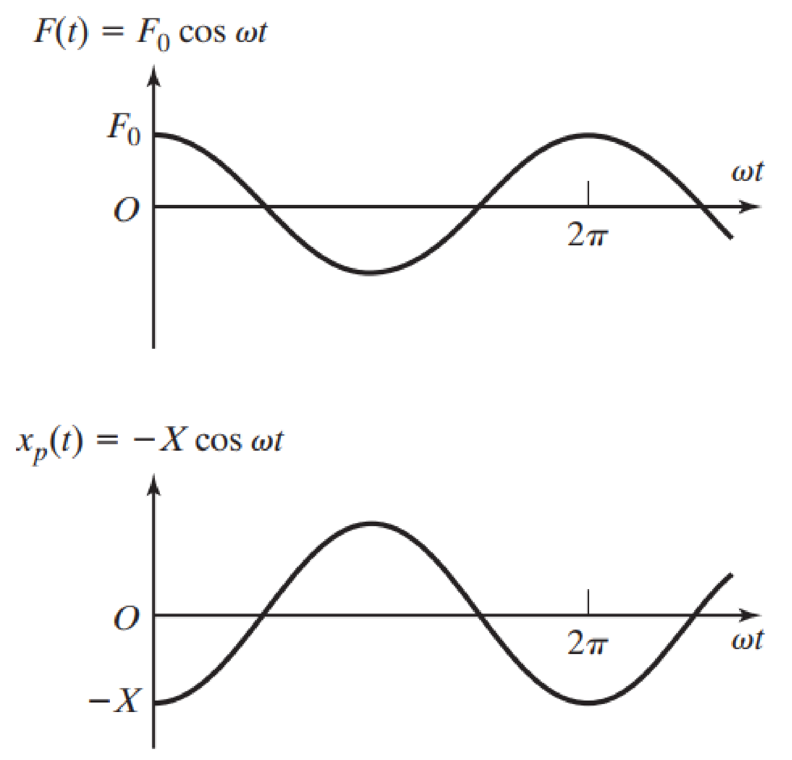
- Case I: $0<\omega/\omega_n<1$ in phase

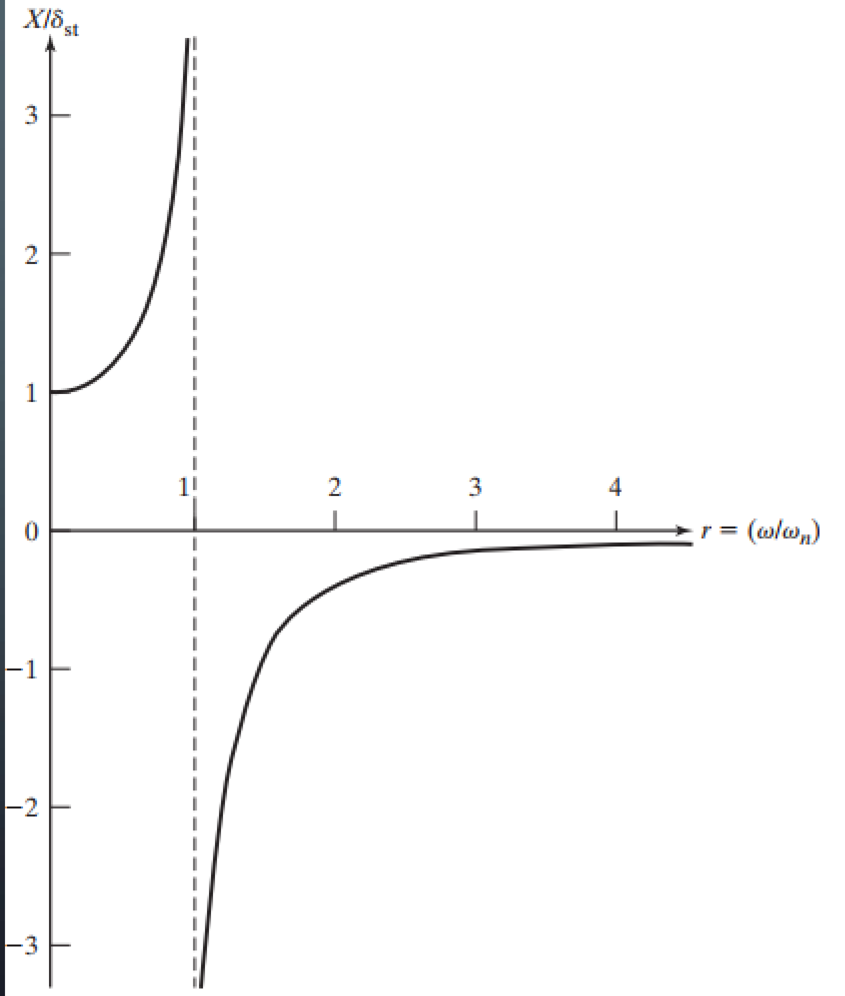




FORCED VIBRATIONS SDOF

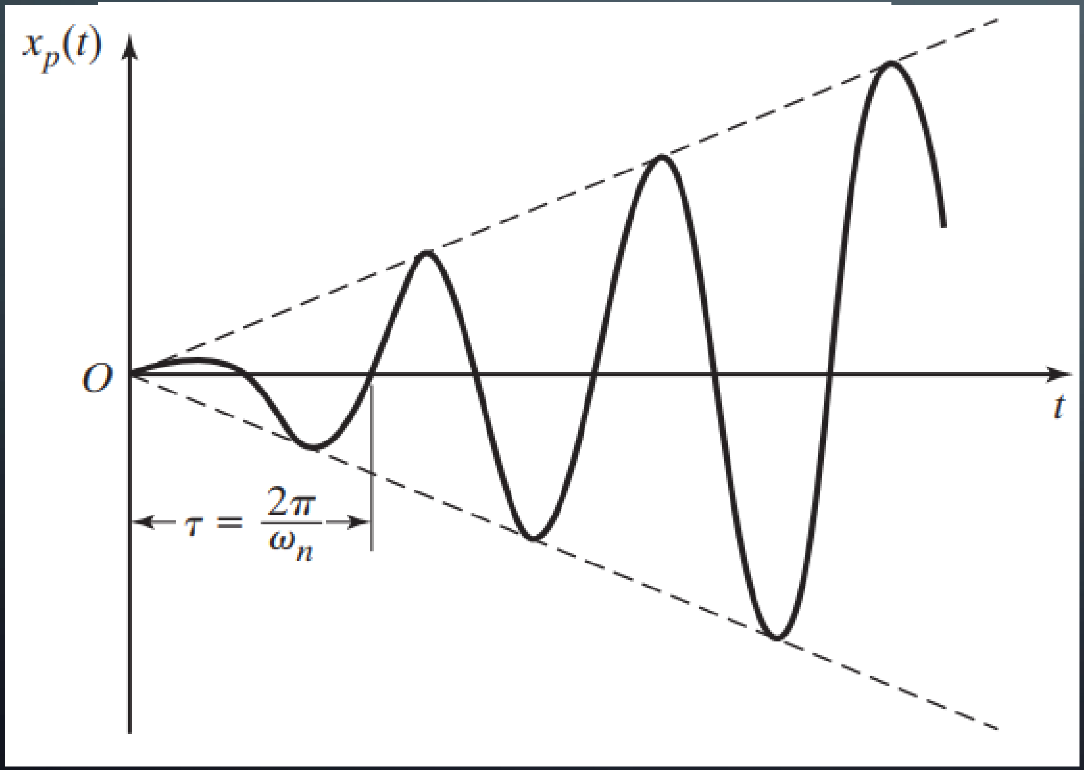
- Case II: ω/ω_n>1 180° out of phase

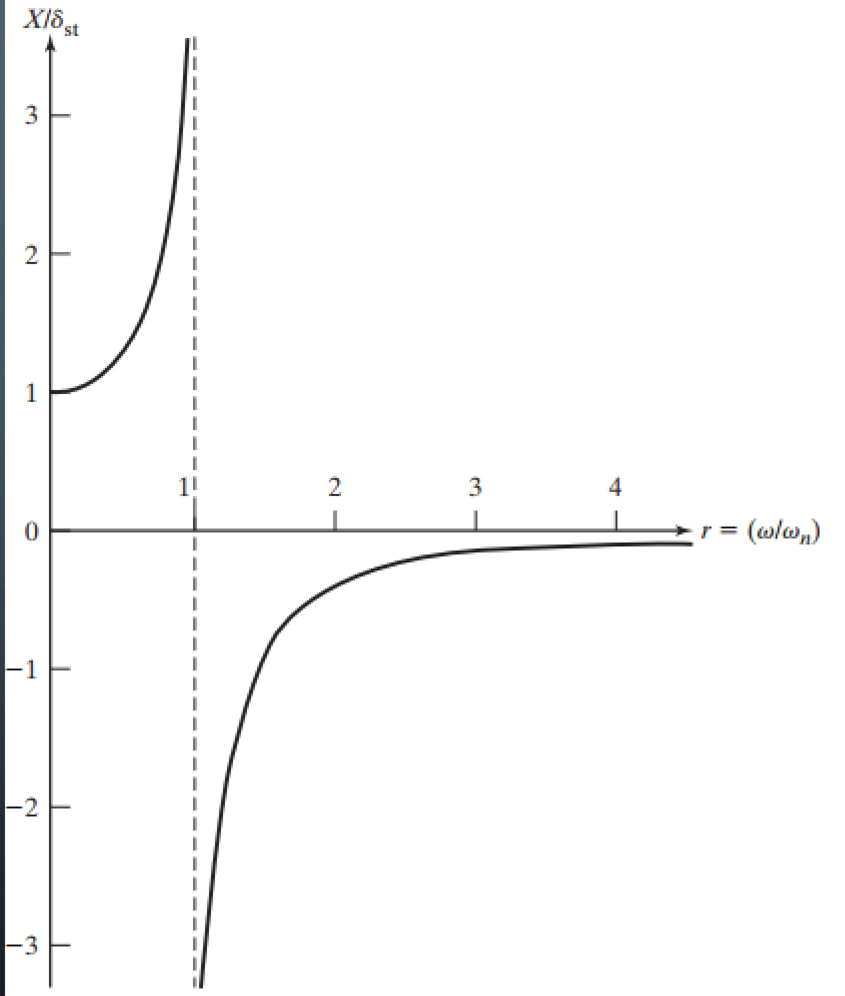




FORCED VIBRATIONS SDOF

- Case III: ω/ω_n=1 resonance





- Damped system

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

- Particular solution

$$x_p(t) = X \cos(\omega t - \phi)$$

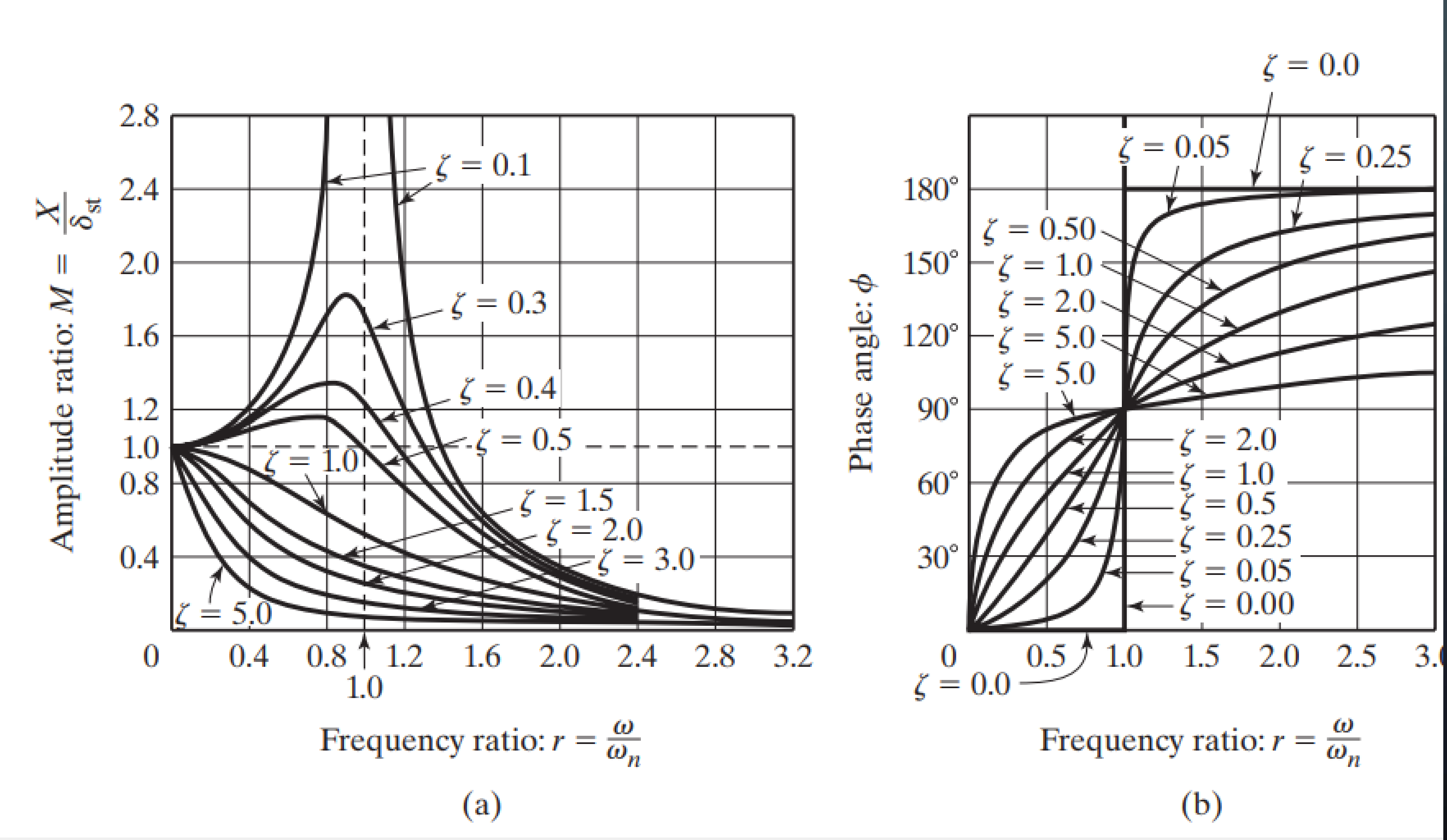
- Steady state solution

$$X = \frac{F_0}{[(k - m\omega^2)^2 + c^2\omega^2]^{1/2}}$$

- Amplification factor & phase angle

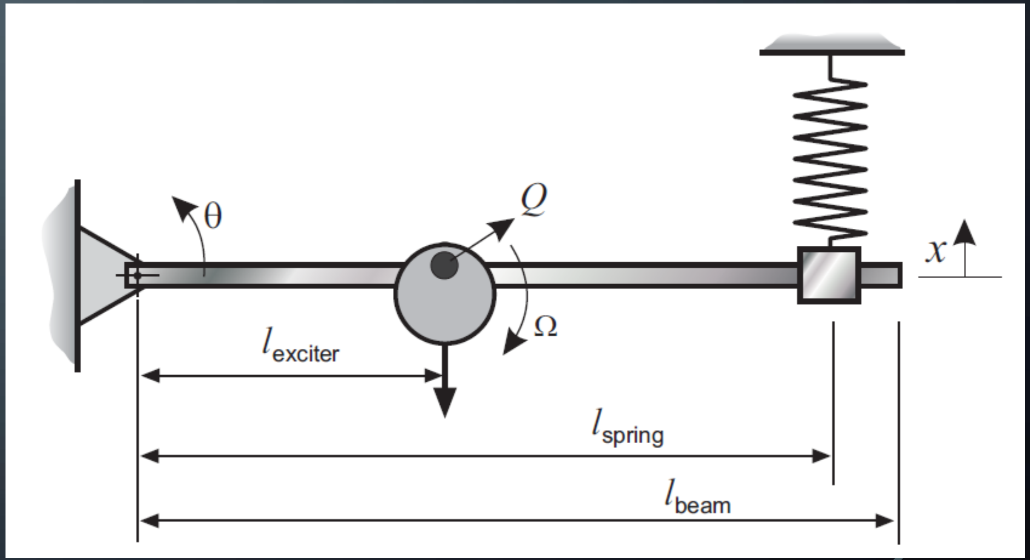
$$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}; \tan \phi = \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

$$x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi)$$
$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$$



FORCED VIBRATIONS OF A BEAM AND SPRING

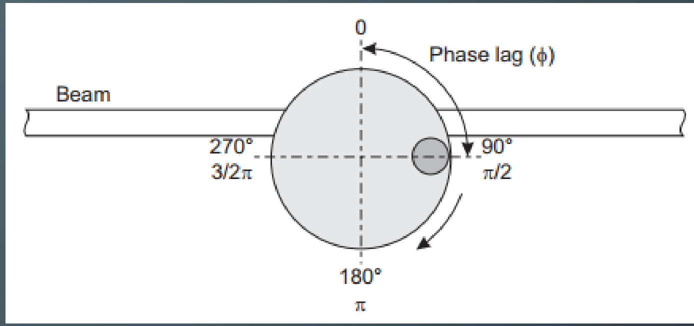
$$I_A\ddot{\theta} + kx_s l_{spring} + cl_{damper}\dot{x}_d = l_{exciter}Q\sin\Omega t$$



$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega^2\theta = \frac{l_{exciter}}{I_A}Q\sin\Omega t$$

$$\theta = \frac{l_{exciter}}{\omega^2 I_A}Q\beta\sin(\Omega t - \phi)$$

FORCED VIBRATIONS OF A BEAM AND SPRING



- Amplification factor

$$\beta = \frac{1}{\sqrt{\left(1-\frac{\Omega^2}{\omega^2}\right)^2+\left(\frac{2\zeta\Omega}{\omega}\right)^2}}$$

- Phase Lag

$$\tan\phi = \frac{\frac{2\zeta\Omega}{\omega}}{1-\frac{\Omega^2}{\omega^2}}$$