MEMS1028
Mechanical Design 1

Lecture 8

Failure (Static loading – ductile material)

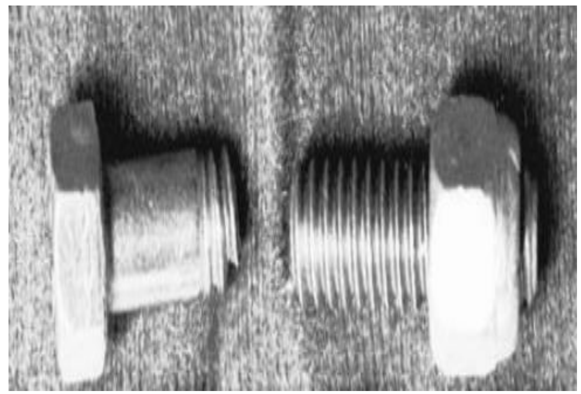
Objectives

- Explain the differences between the failure of ductile and brittle materials
- Apply static failure theories in engineering design involving ductile materials

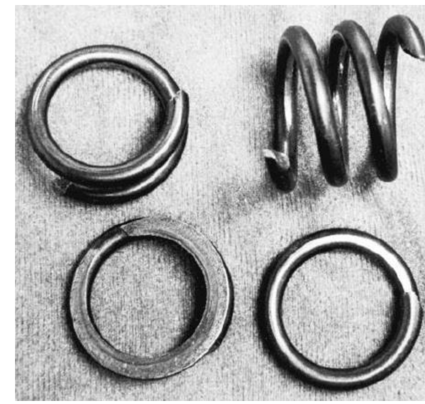
Failure examples

Corrosion failure

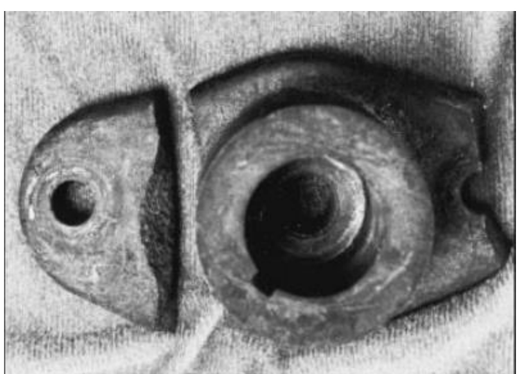
Loading failure



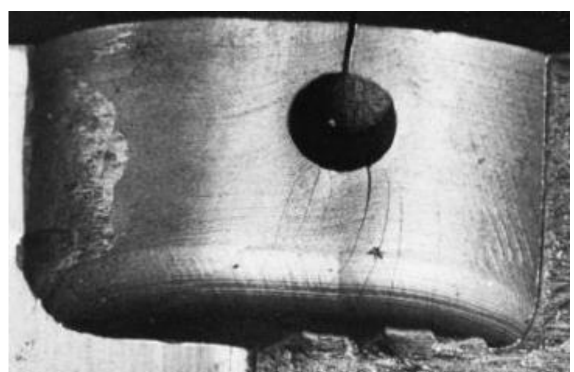
Shear failure



Impact failure



Stress concentration failure



Fatigue failure

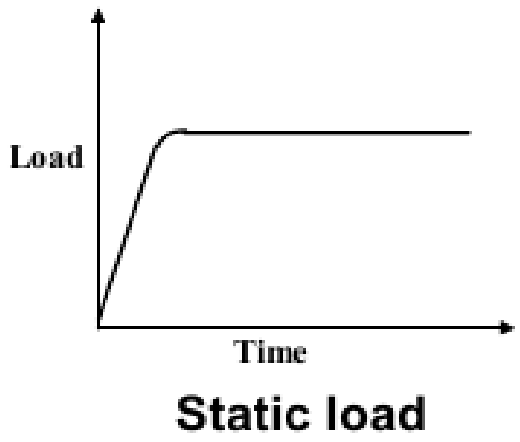
Failure by fracture

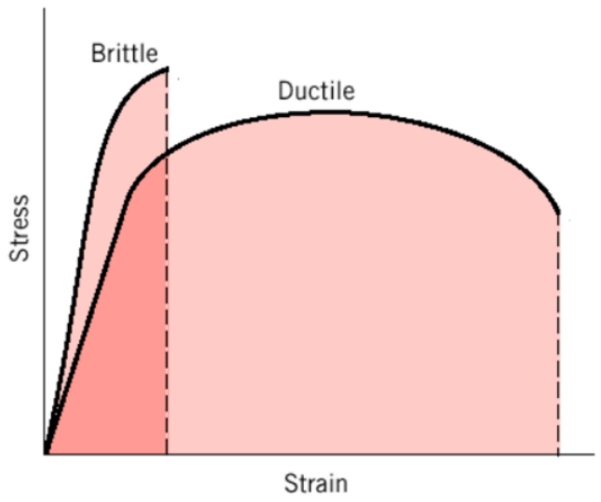
- Why do parts fail by fracture?
  - The simple answer is that parts fail because the applied stresses exceeds the material's strength
- But strength is an inherent property of a material under specific loading conditions and types of strength can include tensile strength, compressive strength, torsional strength etc.
- Then, what kind of stresses cause failure?
  - Remember under any applied load, there is always a combination of normal and shearing stresses in the material
  - Furthermore, strength is a scalar value and stress is a vector

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

We must introduce failure theories that convert the stresses into scalar equivalent for comparison to strength

Loading and fracture

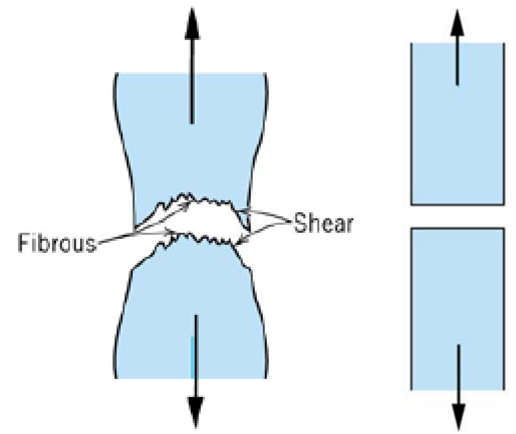




- Ductility is the degree to which a material will deform before ultimate fracture
- Percent elongation is used as a measure of ductility
- Ductile Materials have percent elongation ≥ 5%
- Brittle Materials have percent elongation < 5%
- For machine members subject to repeated or shock or impact loads, materials with percent elongation > 12% are recommended

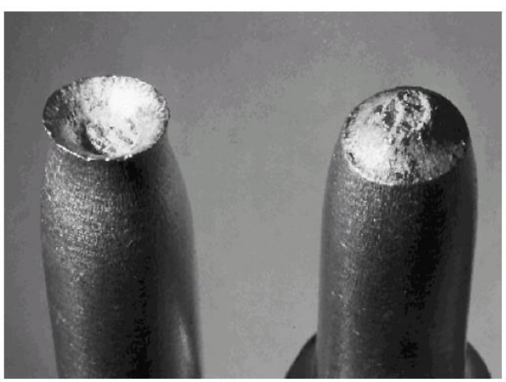
Ductile vs brittle fracture

1) Ductile fracture is generally "cup and cone" and is desirable!
2) Brittle failure gives no warning

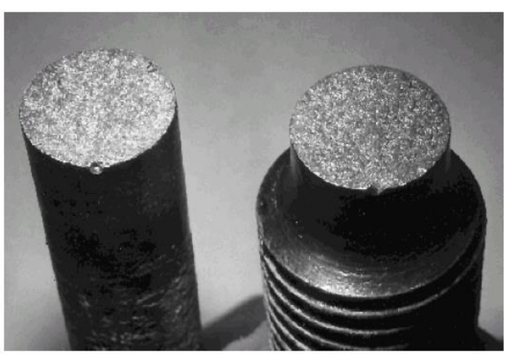


Highly ductile | Moderate ductile | Brittle

Ductile Fracture



(Cap-and-cone fracture in Al



Brittle fracture in a mild steel

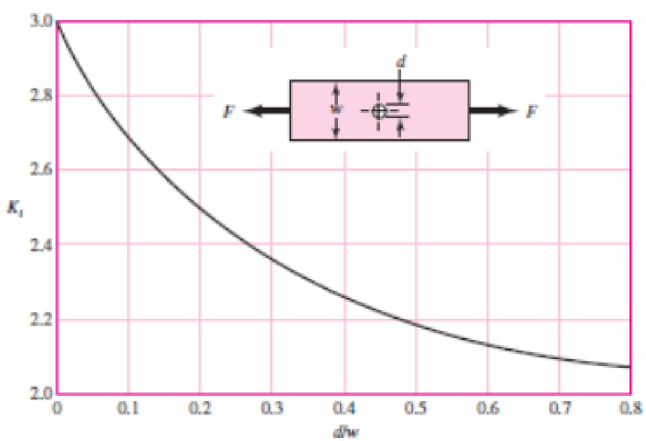
Stress concentration & fracture

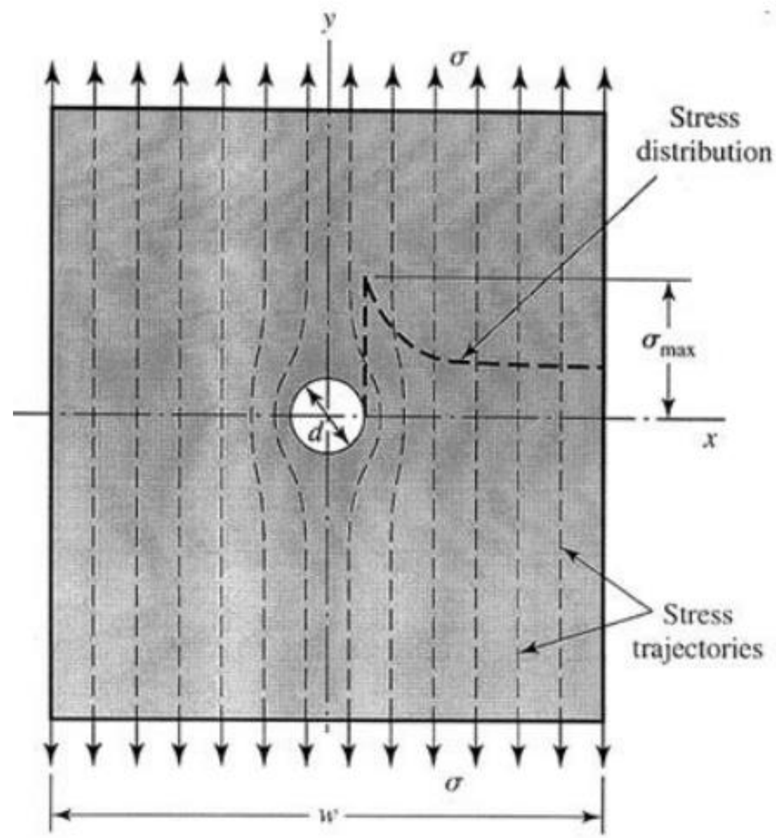
- Localized increase of stress occurs near discontinuities
- K_t is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{max}}{\tau_0}$$

Figure A-15-1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.



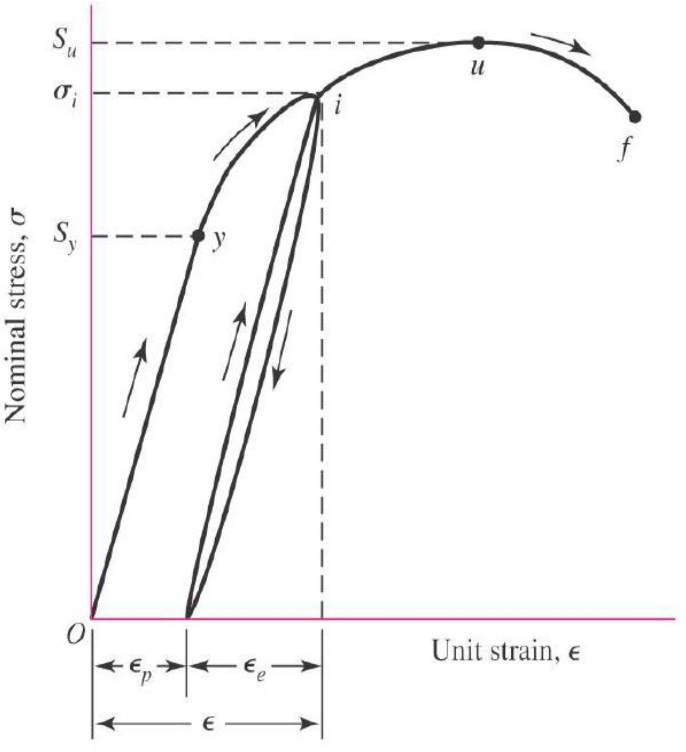


Stress concentration & fracture

With static loads and ductile materials

- Extensive plastic deformation ahead of crack restricts propagation unless applied stress is increased
- These are mainly localized at microscopic level
- Ductile materials can absorb energy before failure
- Overall part does not see damage unless ultimate strength is exceeded

❖ Stress concentration effect is commonly ignored on ductile materials for static loads (but must be included for dynamic loading)

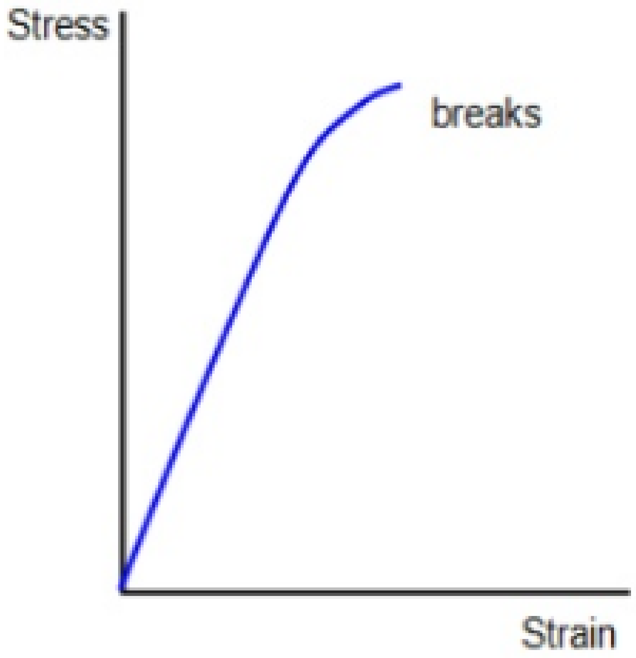


Stress concentration & fracture

For brittle materials

- Very little plastic deformation and crack propagates rapidly without increased in applied stress
- Low energy absorption before failure

❖ Stress concentration must be included on brittle materials for static loads, since localized yielding may quickly result in catastrophic failure



Static failure theories

Failure theories are used to predict if failure would occur under any given state of stress

The generally accepted theories are:

1) Ductile materials (yield criteria)

- ❖ Maximum shear stress (MSS) – Tresca Criterion
- ❖ Distortion energy (DE) – von-Mises Criterion
- ❖ Ductile Coulomb-Mohr (DCM)

2) Brittle materials (fracture criteria)

- Maximum normal stress (MNS),
- Brittle Coulomb-Mohr (BCM),
- Modified Mohr (MM),

Maximum shear stress theory

- Theory: Yielding begins when the maximum shear stress in a stress element exceeds the maximum shear stress in a tension test specimen
- Let S_y be the material yield strength
- At yielding, the maximum shear stress is $S_y/2$
- The theory could be restated as: Yielding begins when the maximum shear stress in a stress element exceeds $S_y/2$
- For any stress element, find the maximum shear stress using Mohr's circle and compare the maximum shear stress to $S_y/2$

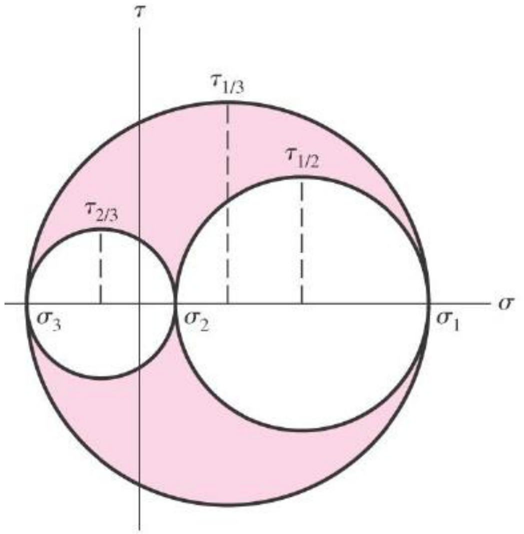
Ordering the principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \text{ or } \sigma_1 - \sigma_3 \geq S_y \text{ (yielding criteria)}$$

Incorporating a design factor "*n*":

$$\tau_{max} = \frac{S_y}{2n} \text{ or } \sigma_1 - \sigma_3 = \frac{S_y}{n} \text{ (design criteria)}$$

Or solving for factor of safety: $n = \frac{S_y/2}{\tau_{max}} = \frac{S_y}{\sigma_1 - \sigma_3}$



Maximum shear stress theory

To simplify, consider a plane stress state (one of the principal stress is zero)
Let σ_A and σ_B represent the two non-zero principal stresses, then order them with the zero principal stress such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$;
Assuming $\sigma_A \geq \sigma_B$ there are three cases to consider:

- Case 1: $\sigma_A \geq \sigma_B \geq 0$ (tensile stresses)

  For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$

  $\sigma_1 - \sigma_3 \geq S_y$ reduces to $\sigma_A \geq S_y$ and $\sigma_A = \frac{S_y}{n}$
- Case 2: $\sigma_A \geq 0 \geq \sigma_B$ (tensile and compressive stresses)

  For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$

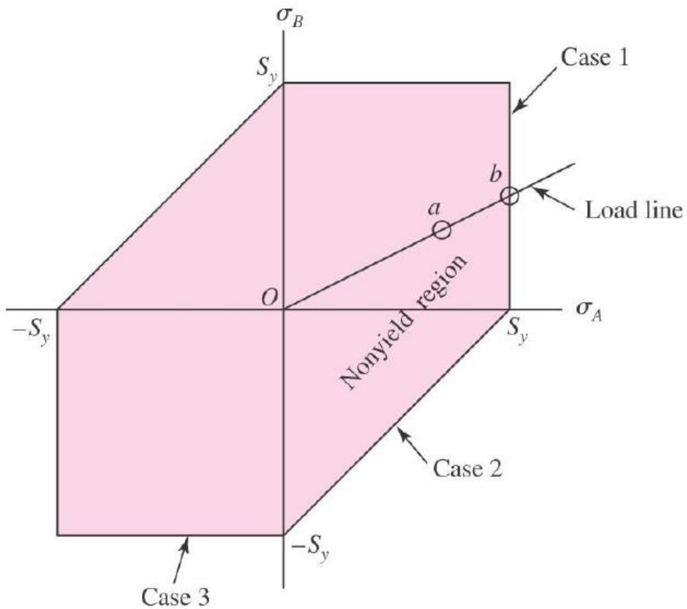
  $\sigma_1 - \sigma_3 \geq S_y$ reduces to $\sigma_A - \sigma_B \geq S_y$ and $\sigma_A - \sigma_B = \frac{S_y}{n}$
- Case 3: $0 \geq \sigma_A \geq \sigma_B$ (compressive stresses)

  For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$

  $\sigma_1 - \sigma_3 \geq S_y$ reduces to $\sigma_B \leq -S_y$ and $\sigma_B = -\frac{S_y}{n}$

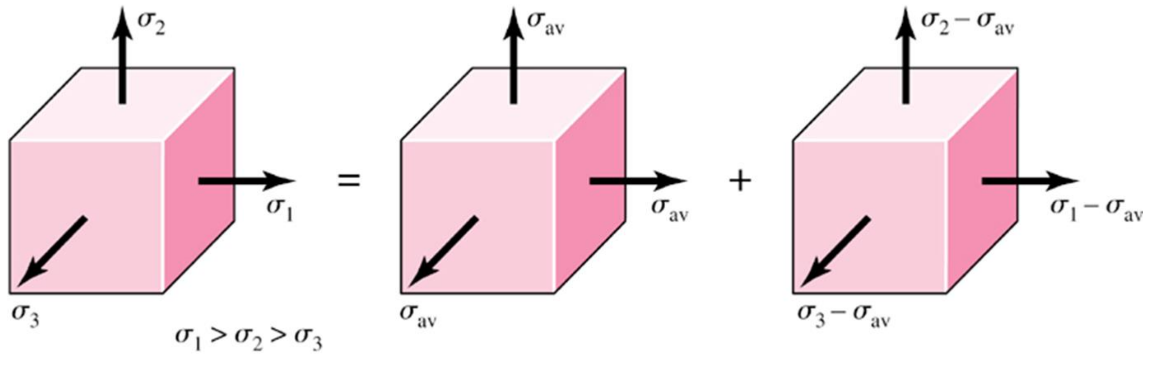
Maximum shear stress theory

- Plot three cases on principal stress axes
- Case 1: $\sigma_A \geq \sigma_B \geq 0$ (tensile stresses)
  $\sigma_A \geq S_y$
- Case 2: $\sigma_A \geq 0 \geq \sigma_B$ (tensile and compressive stresses)
  $\sigma_A - \sigma_B \geq S_y$
- Case 3: $0 \geq \sigma_A \geq \sigma_B$ (compressive stresses)
  $\sigma_B \leq -S_y$
- Other lines are symmetric cases
- Inside envelope is predicted safe zone
- Commonly used for design situations
- Conservative in all quadrants
- Close match with experimental data

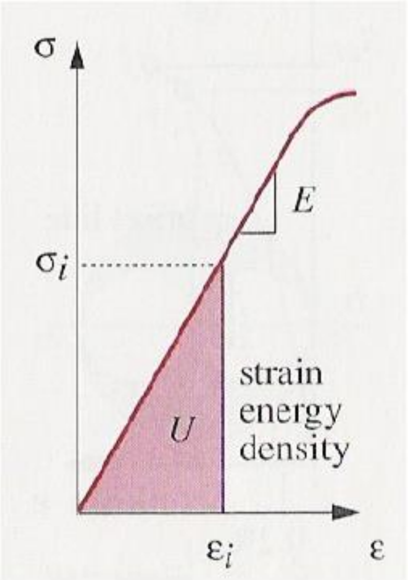


Distortion energy theory

- Also known as: Octahedral Shear Stress; Shear Energy; Von Mises;
- Theorizes that if strain energy is divided into hydrostatic volume changing energy and angular distortion energy, the yielding is primarily affected by the distortion energy



(a) Triaxial stresses (b) Hydrostatic component (c) Distortional component



Theory: Yielding occurs when the distortion strain energy per unit volume reaches the distortion strain energy per unit volume for yield in simple tension or compression of the same material

Distortion energy theory

- Failure occurs if distortion energy exceeds distortion energy of tension test specimen (note: σ_1, σ_2, and σ_3, are principal stresses):

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y$$

- Left hand side is defined as von Mises stress σ'
- In terms of *xyz* components, in three dimensions

$$\sigma' = \frac{1}{\sqrt{2}}\left[\left(\sigma_x-\sigma_y\right)^2+\left(\sigma_y-\sigma_z\right)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)\right]^{1/2}$$

- For plane stress (in terms of *xyz* components):

$$\sigma' = \left(\sigma_x^2-\sigma_x\sigma_y+\sigma_y^2+3\tau_{xy}^2\right)^{1/2}$$

- For plane stress (in terms of principal stress components σ_A, and σ_B):

$$\sigma' = [\sigma_A^2-\sigma_A\sigma_B+\sigma_B^2]$$

Distortion energy theory

- Von Mises Stress can be thought of as a single, equivalent, or effective stress for the entire general state of stress in a stress element
- Distortion Energy failure theory simply compares von Mises stress to yield strength

$$\sigma' \geq S_y$$

- Introducing a design factor,

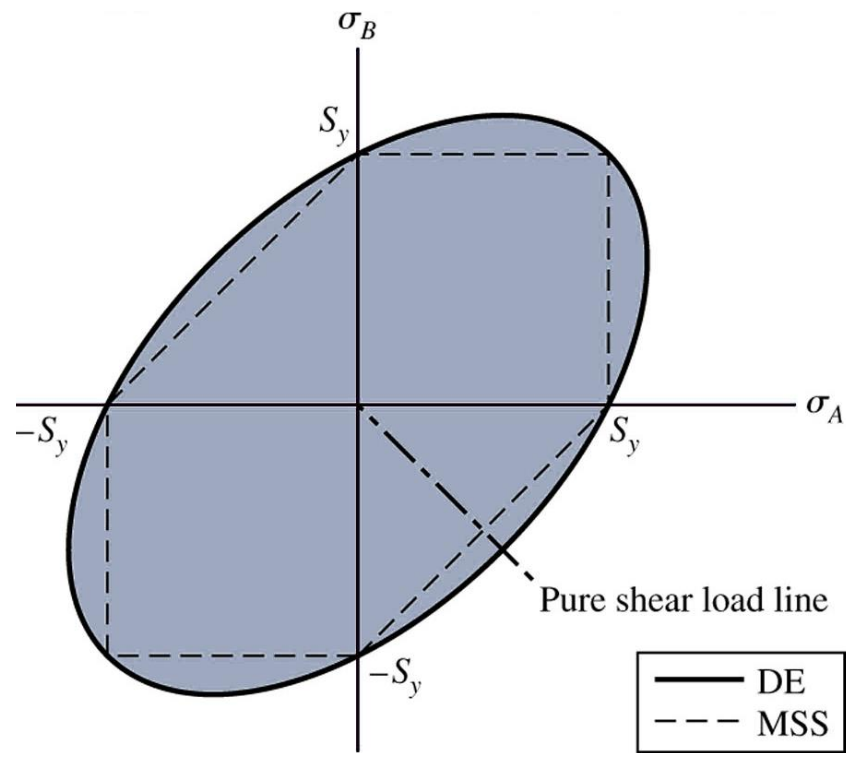
$$\sigma' = \frac{S_y}{n}$$

- Expressing as factor of safety,

$$n = \frac{S_y}{\sigma'}$$

Distortion energy theory

- Criterion for the distortion energy theory in 2-D is an equation for an ellipse
- Distortion Energy curve typically equates to about 50% reliability from a design perspective
- Commonly used for analysis situations
- The maximum shear stress theory falls inside the distortion energy theory. Maximum Shear Stress theory useful for design situations where higher reliability is desired



Intersection of pure shear load line $\sigma_A = -\sigma_B = \tau$ with failure curve

Example 1

A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 700$MPa and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states based on the maximum shear stress and distortion energy theories:
(a) 490, 490, 0 MPa
(b) 210, 490, 0 MPa
(c) 0, 490, -210 MPa
(d) 0, -210, -490 MPa
(e) 210, 210, 210 MPa

- Note: True strain = 0.55 = ln(1+strain),
- Percent elongation ≥ 5% is considered ductile
- Given $S_{yt} = S_{yc} = 700$MPa (equal strength in compression & tension)

Example 1

Yield strength of $S_{yt} = S_{yc} = 700$MPa

(a) 490, 490, 0 MPa

- MSS: Case 1: $\sigma_A \geq \sigma_B \geq 0$ (tensile stresses)

  For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$ and $n = \frac{S_y}{\sigma_A} = 1.43$

- DE: $\sigma' = [\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2] = 490$MPa and $n = \frac{S_y}{\sigma'} = 1.43$

(b) 210, 490, 0 MPa

- MSS: Case 1: $\sigma_A \geq \sigma_B \geq 0$ (tensile stresses)

  For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$ and $n = \frac{S_y}{\sigma_A} = 1.43$

- DE: $\sigma' = [\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2] = 426$MPa and $n = \frac{S_y}{\sigma'} = 1.64$

Example 1

Yield strength of $S_{yt} = S_{yc} = 700$MPa

(c) 0, 490, -210 MPa

- MSS: Case 2: $\sigma_A \geq 0 \geq \sigma_B$ (tensile and compressive stresses)

  For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$ and $n = \frac{S_y}{\sigma_A - \sigma_B} = 1$
- DE: $\sigma' = [\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2] = 622\text{MPa}$ and $n = \frac{S_y}{\sigma'} = 1.13$

(d) 0, -210, -490 MPa

- MSS: Case 3: $0 \geq \sigma_A \geq \sigma_B$ (compressive stresses)

  For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$ and $n = -\frac{S_y}{\sigma_B} = 1.43$
- DE: $\sigma' = [\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2] = 426\text{MPa}$ and $n = \frac{S_y}{\sigma'} = 1.64$

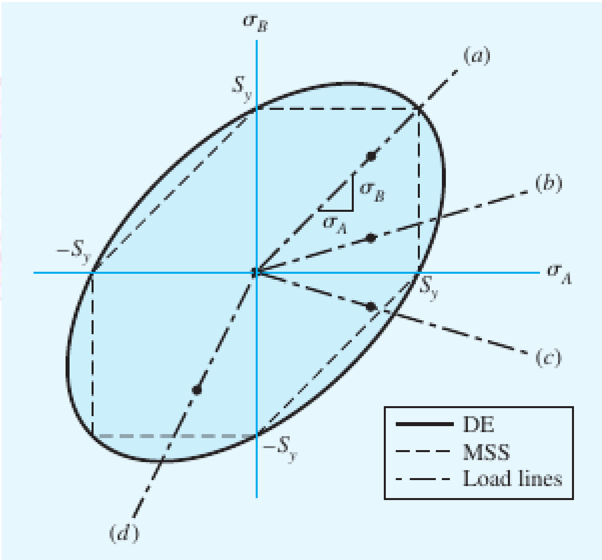
Example 1

Yield strength of $S_{yt} = S_{yc} = 700\text{MPa}$

(e) 210, 210, 210 MPa

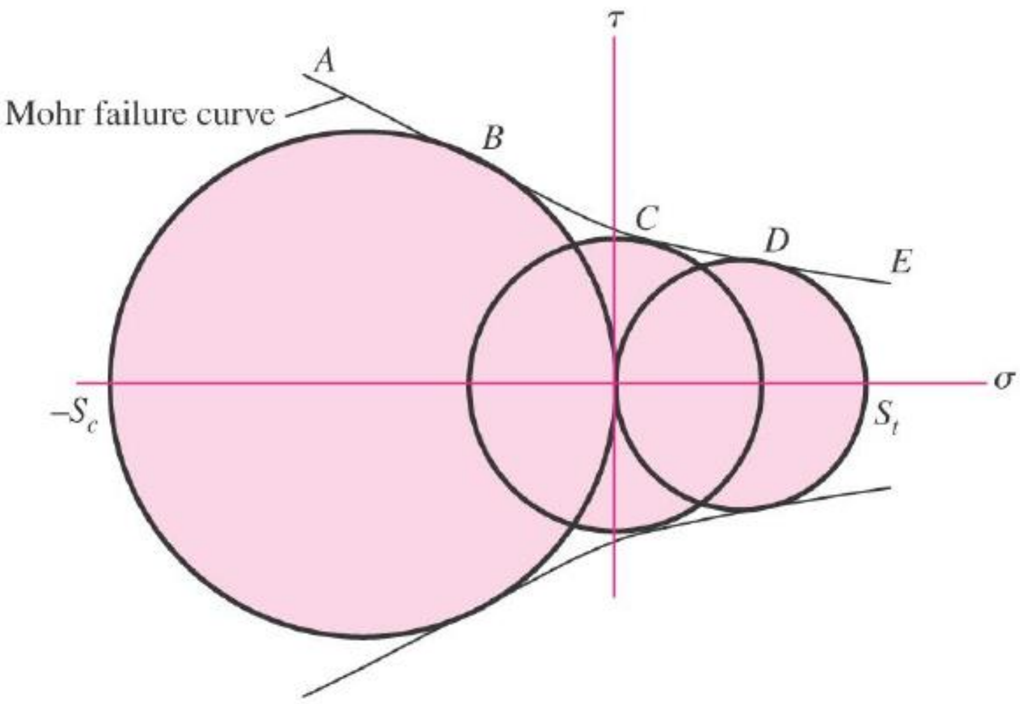
- MSS: Not a plane stress case and $\sigma_3 = \sigma = 210\text{MPa}$ and $n = \frac{S_y/2}{\tau_{max}} = \frac{S_y}{\sigma_1 - \sigma_3} \rightarrow \infty$
- DE: $\sigma' = \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} = 0$ and $n = \frac{S_y}{\sigma'} \rightarrow \infty$

	(a)	(b)	(c)	(d)	(e)
DE	1.43	1.64	1.13	1.64	∞
MSS	1.43	1.43	1.00	1.43	∞



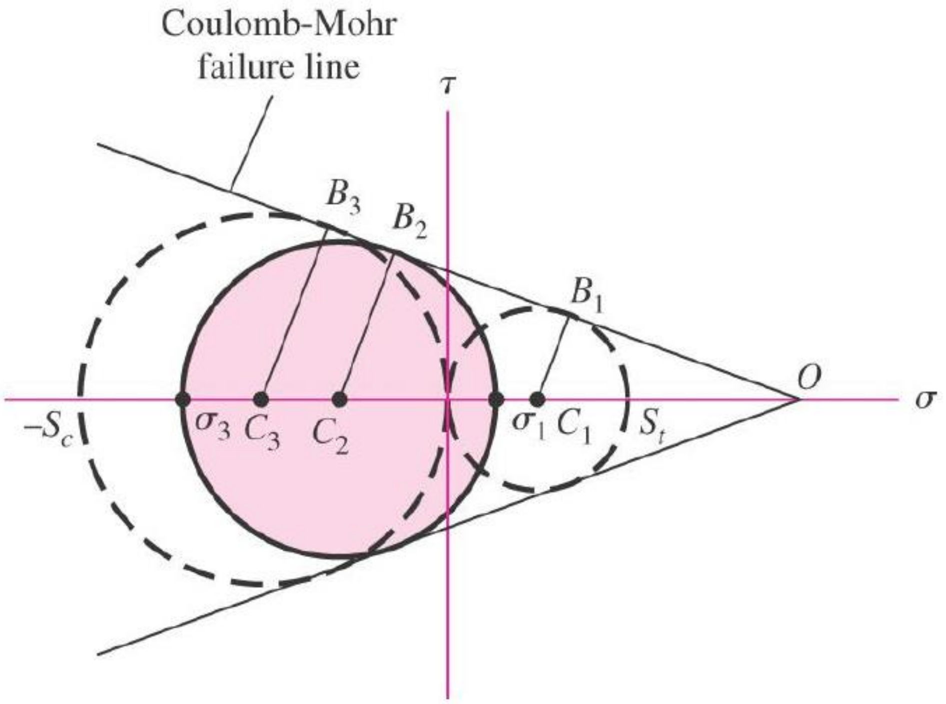
Mohr theory

- Some materials have compressive strengths different from tensile strengths
- Mohr theory is based on three simple tests: tension, compression, and shear
- Plotting Mohr's circle for each, bounding curve defines failure envelope



Coulomb-Mohr theory

- ❖ Curved failure curve is difficult to determine analytically
- ❖ Coulomb-Mohr theory simplifies to linear failure envelope using only tension and compression tests (dashed circles)



From the geometry, derive the failure criteria:

$$\frac{B_2C_2 - B_1C_1}{OC_2 - OC_1} = \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1}$$

$$\frac{B_2C_2 - B_1C_1}{C_1C_2} = \frac{B_3C_3 - B_1C_1}{C_1C_3}$$

$$B_1C_1 = \frac{S_t}{2}; \qquad B_3C_3 = \frac{S_c}{2};$$

$$B_2C_2 = \frac{(\sigma_1 - \sigma_3)}{2};$$

Coulomb-Mohr theory

Failure criteria simplify to:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$$

- To plot on principal stress axes, consider three cases:
- Case 1: $\sigma_A \geq \sigma_B \geq 0$

For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$

$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$ reduces to $\sigma_A \geq S_t$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$

For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$

$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$ reduces to

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1$$

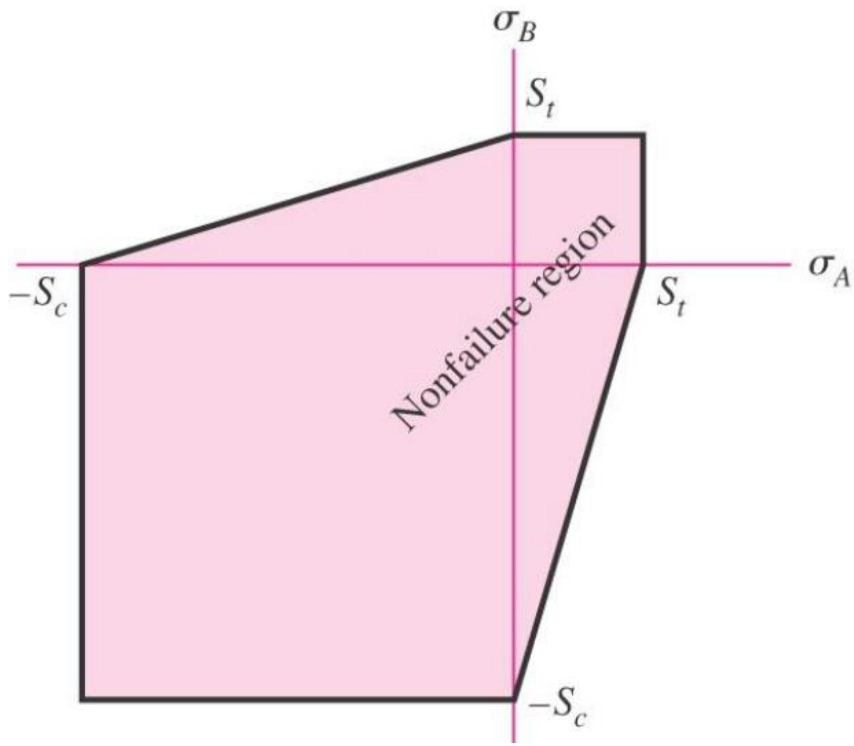
- Case 3: $0 \geq \sigma_A \geq \sigma_B$

For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$

$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1$ reduces to $\sigma_B \leq -S_c$

Coulomb-Mohr theory

- ❖ Plot three cases on principal stress axes
- ❖ Similar to Maximum shear stress theory, except with different strengths for compression and tension



- Case 1: $\sigma_A \geq \sigma_B \geq 0$

  $$\sigma_A \geq S_t$$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$

  $$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1$$

- Case 3: $0 \geq \sigma_A \geq \sigma_B$

  $$\sigma_B \leq -S_c$$

Coulomb-Mohr theory

- Incorporating factor of safety:

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n}$$

- For ductile material, use tensile and compressive yield strengths
- For brittle material, use tensile and compressive ultimate strengths
- Since failure line is a function of tensile and compressive strengths, shear yield strength is also a function of the tensile and compressive yield strengths:

$$S_{sy} = \frac{S_t S_c}{S_t + S_c}$$

Example 2

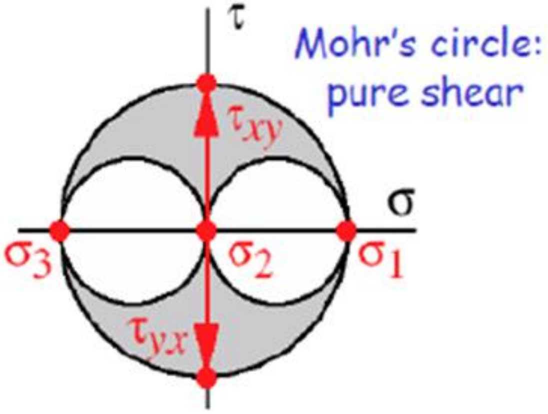
A 25-mm-diameter shaft is statically torqued to 230Nm. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

For a solid shaft subjected to torsion, the maximum shear stress is given by

$$\tau = \frac{Tr}{J} = \frac{32T}{\pi d^3} = 75\text{MPa}$$

Given $S_{yt} \neq S_{yc}$ (unequal strength in compression & tension), we should apply Coulomb-Mohr theory

- For pure torsion, two non-zero principal stresses are 75MPa & -75MPa (see Moh's circle)



Example 2

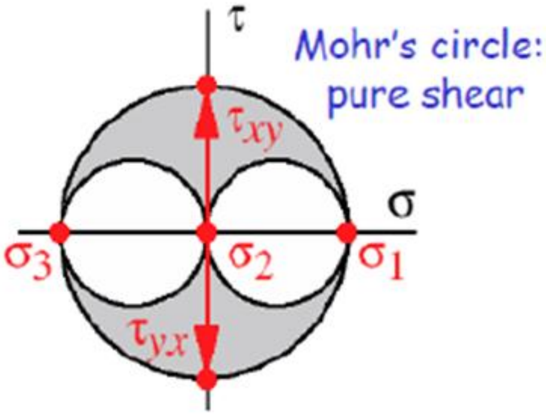
Given S_{yt} = 160MPa and S_{yc} = 170MPa.
The two principal stresses are σ_1 = 170MPa and σ_3 = -75MPa

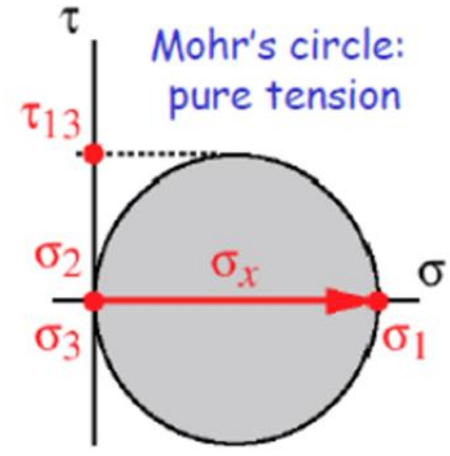
The factor of safety can be found using 2 approaches:

1) $\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n}$ and n = 1.1

2) Find $S_{sy} = \frac{S_t S_c}{S_t + S_c} = 82.4\text{MPa}$; $\tau_{\max} = 75\text{MPa}$ (refer to Mohr's circle)

Factor of safety $n = \frac{S_{sy}}{\tau_{max}} = 1.1$





Summary – ductile failure

- Either the maximum-shear-stress theory or the distortion-energy theory is acceptable for design and analysis of materials that would fail in a ductile manner.
- For design purposes the maximum-shear-stress theory is easy, quick to use, and conservative.
- If the problem is to learn why a part failed, then the distortion-energy theory may be the best to use.
- For ductile materials with unequal yield strengths, σ_{yt} in tension and σ_{yc} in compression, the Mohr theory is the best available.