MEMS1028
Mechanical Design 1

Lecture 7

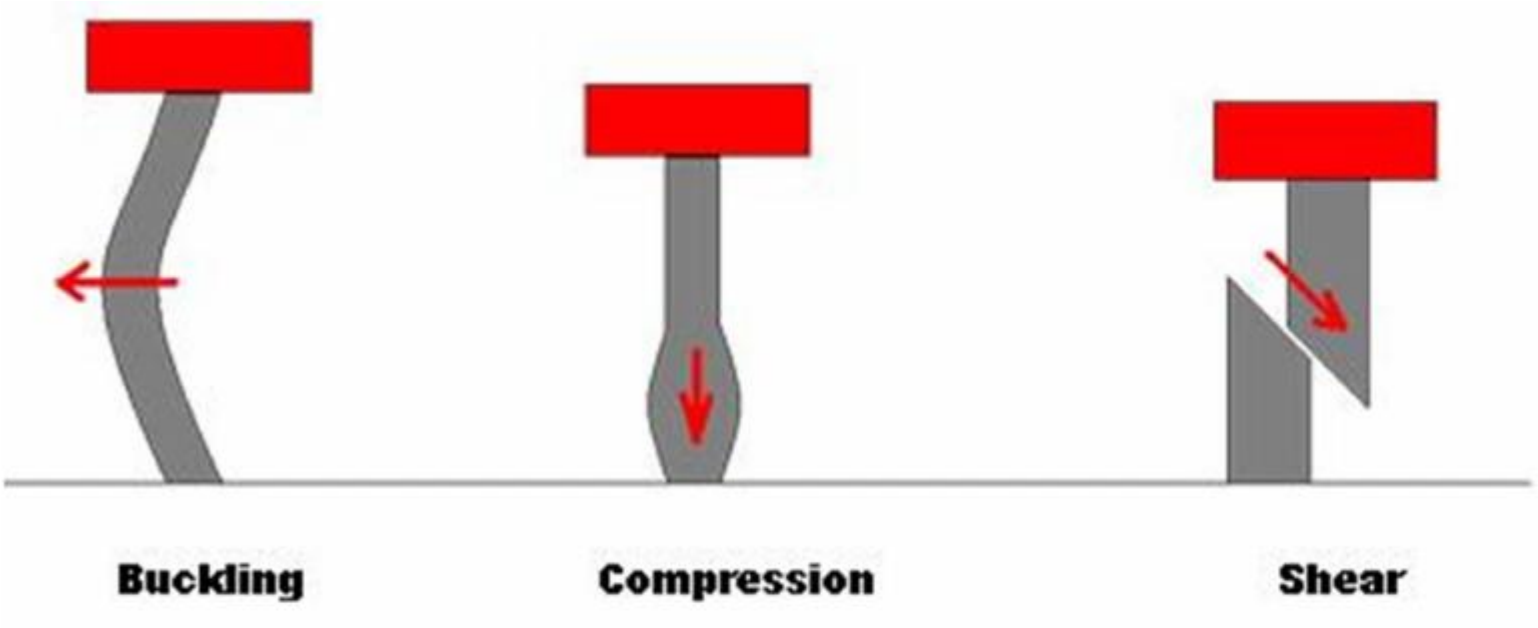
Advanced deformation analysis (Columns)

Objectives

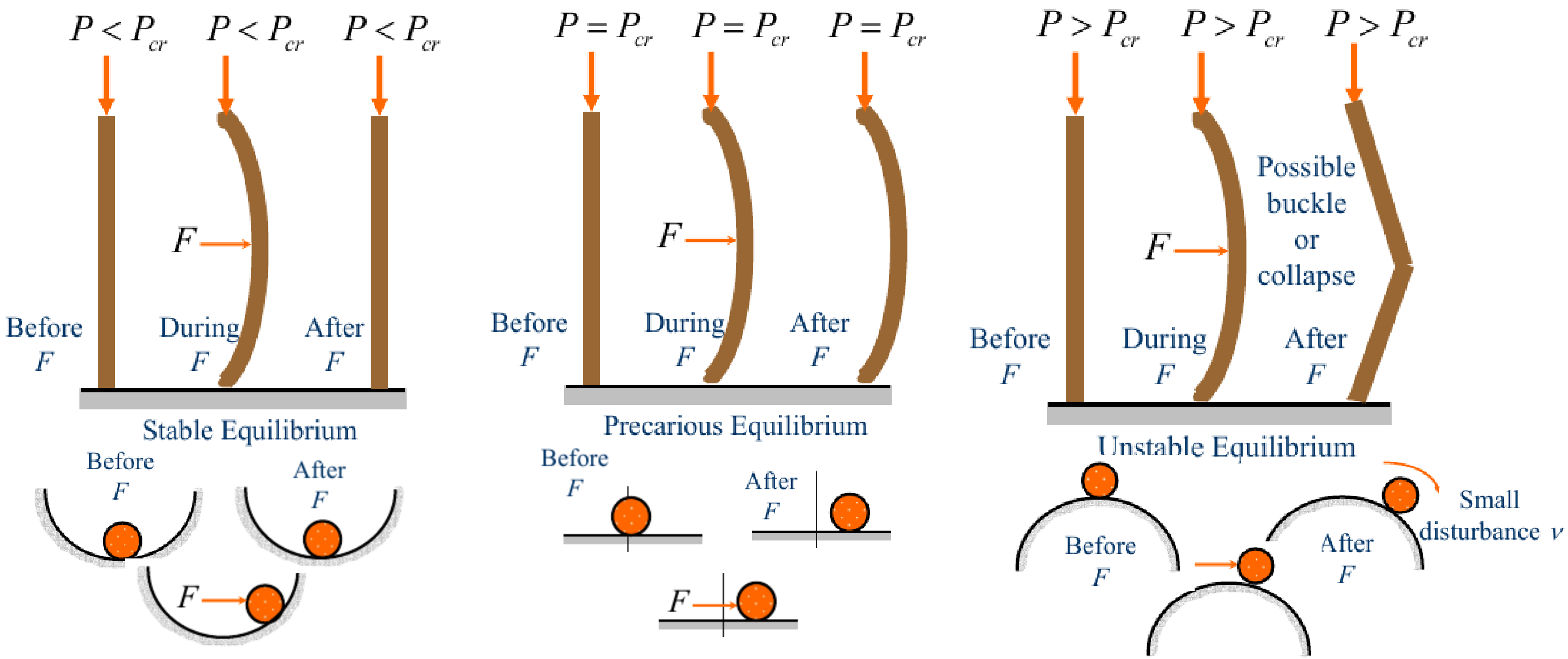
- Explain the concept of buckling
- Design column based on the buckling critical loads
- Analyze eccentric loading in the design of columns
- Design of elements based on impact loadings

Column design

A column is a long, slender member that carries an axial compressive load and that fails due to buckling rather than due to failure of the material of the column.



Stability of Columns



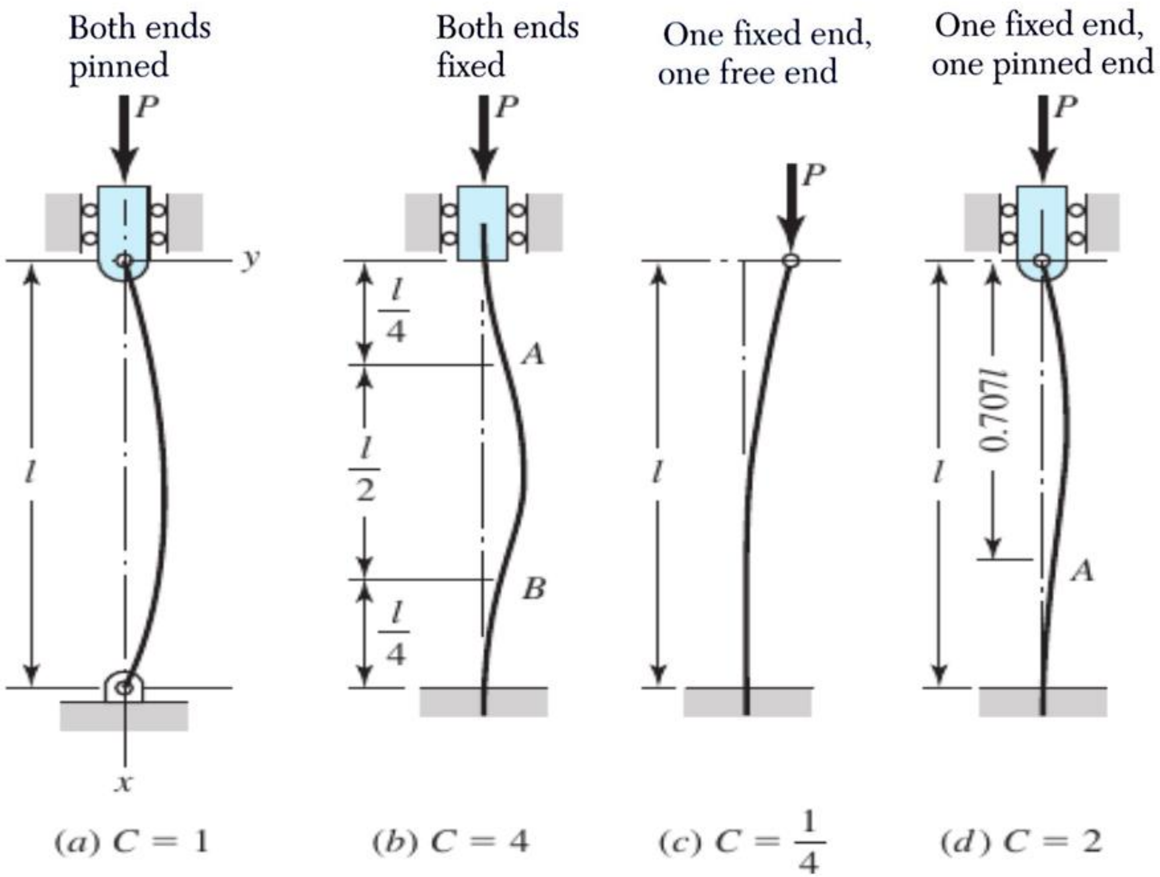
For $P \geq P_{cr}$

- The column either will remain in the bent position or will completely collapse and fracture
- For axial loads greater than P_{cr} the column is one of unstable equilibrium in that a small disturbance will tend to grow into an excessive deformation

Failure due to buckling

Long columns – central loading

- ❖ Force P shown acts along the centroidal axis of the column



- When P reaches a critical value, the column becomes unstable
- Critical load depends on the boundary (or end) conditions:

$$P_{cr} = \frac{C\pi^2 EI}{l^2}$$

Long columns – central loading

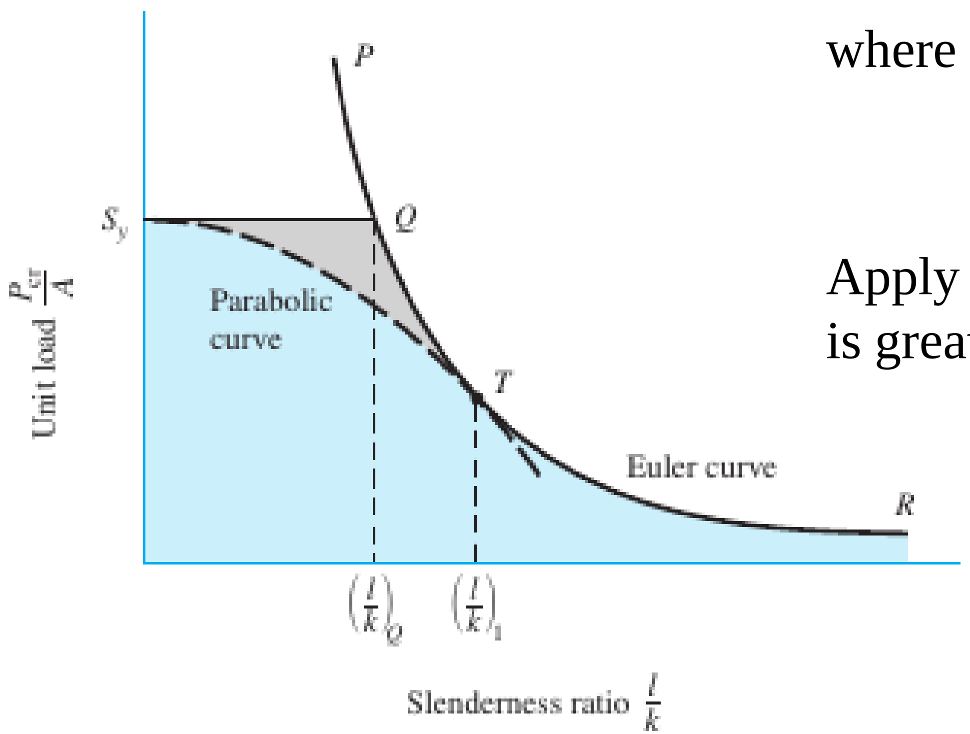
$$P_{cr} = \frac{C\pi^2 EI}{l^2} \quad \text{can be rewritten as} \quad \frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$$

- Where area moment of inertia $I = Ak^2$
- (P_{cr}/A) = critical unit load
- A = Area and k = radius of gyration
- (l/k) = slenderness ratio

Column End Conditions	End-Condition Constant C Theoretical Value	Conservative Value	Recommended Value*
Fixed-free	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
Rounded-rounded	1	1	1
Fixed-rounded	2	1	1.2
Fixed-fixed	4	1	1.2

*To be used only with liberal factors of safety when the column load is accurately known.

Long columns – central loading



When to apply $\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$

Most designers select point *T* such that

$$(P_{cr}/A) = \left(S_y/2\right)$$

where

$$(l/k)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2}$$

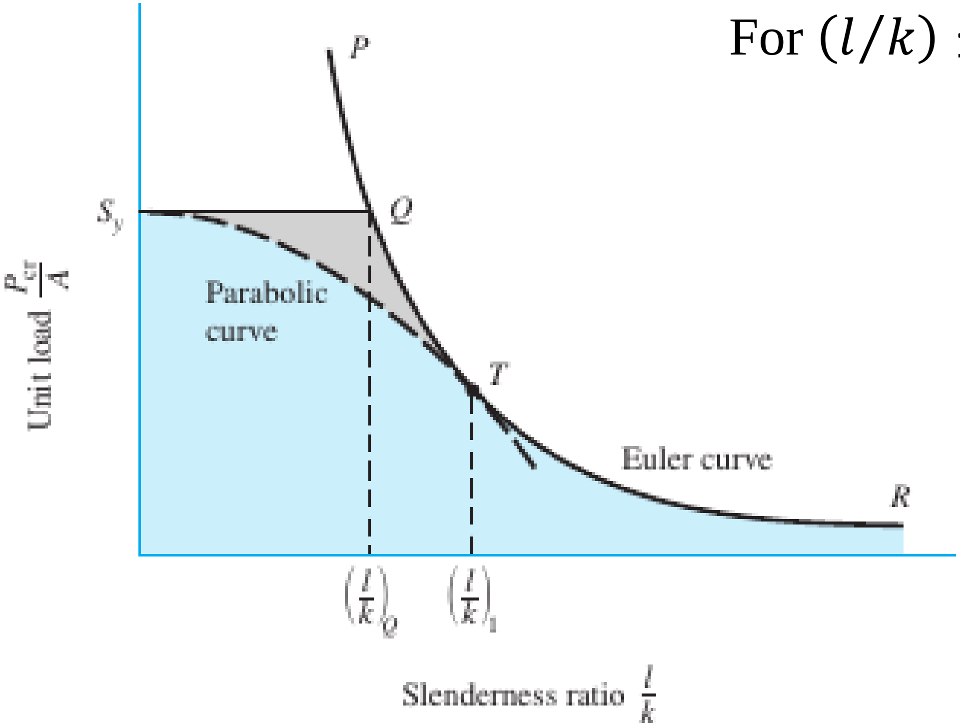
Apply Euler equation if slenderness ratio is greater than $(l/k)_1$

Intermediate-length Columns – central loading

Use the parabolic or J. B. Johnson formula if slenderness ratio is equal or less than $(l/k)_1$

For $(l/k) \leq (l/k)_1$ apply

$$\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{l}{k}\right)^2 \frac{1}{CE}$$



Analysis of columns

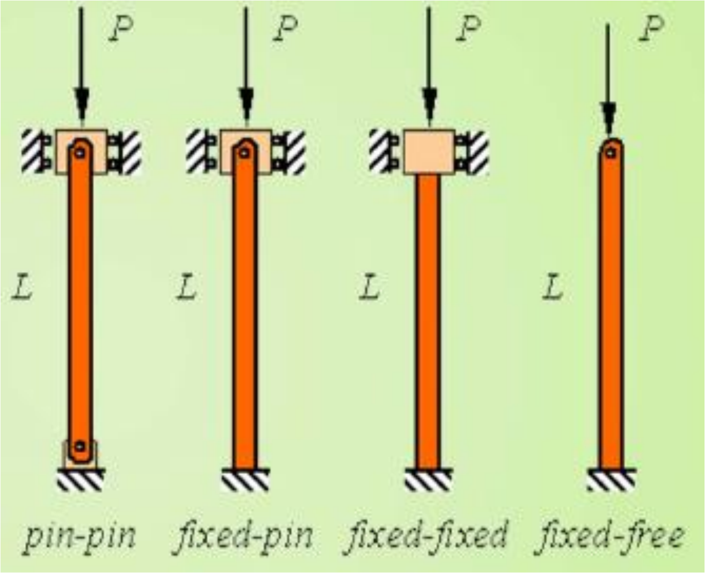
MATERIALS:

1) Modulus of elasticity (E)
2) Yield strength (S_y)

PROPERTIES OF CROSS SECTIONS:

1) Area A
2) Moment of inertia I
3) Radius of gyration $k = \sqrt{\frac{I}{A}}$

CONNECTIONS:



$C = 1$ $C = 2$ $C = 4$ $C = 1/4$

COLUMN TYPES:

1) Slenderness ratio (l/k)
2) Transition slender ratio $(l/k)_1 = \sqrt{\left(\frac{2\pi^2 CE}{S_y}\right)}$

If $(l/k) \leq (l/k)_1$ then column is short (use Johnson formula

If $(l/k) > (l/k)_1$ then column is long (use Euler formula

Example 1

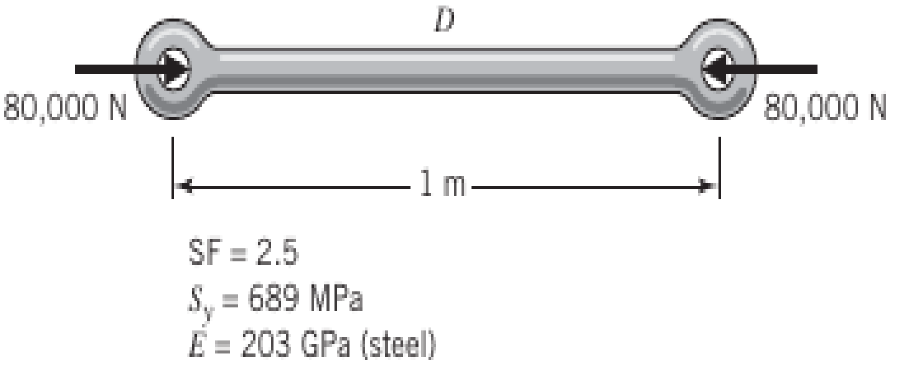
An industrial machine requires a solid, round connecting rod 1m long (between pinned ends) that is subjected to a maximum compressive force of 80kN. Using a safety factor of 2.5, what diameter is required if steel is used, having properties of S_y = 689MPa, E = 203GPa?

Note: $C = 1$ for pinned ends;
Design overload P = (fs)80kN;
where (fs) = 2.5
P = 200kN
For a round rod, area $A = \pi r^2$;

$$I = (1/4)\pi r^4$$

Radius of gyration $k = \sqrt{(I/A)}$

$$k = r/2$$



Example 1

Assume Euler's equation is valid:

$$\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$$

$$\frac{200(10^3)}{\pi r^2} = \frac{\pi^2(203)(10^9)}{(2/r)^2}$$

$$r^4 = \frac{200(10^3)4}{(203)(10^9)\pi^3}$$

Radius r =0.0189m or diameter d = 0.0378m

Radius of gyration k = r/2 = 0.00945m

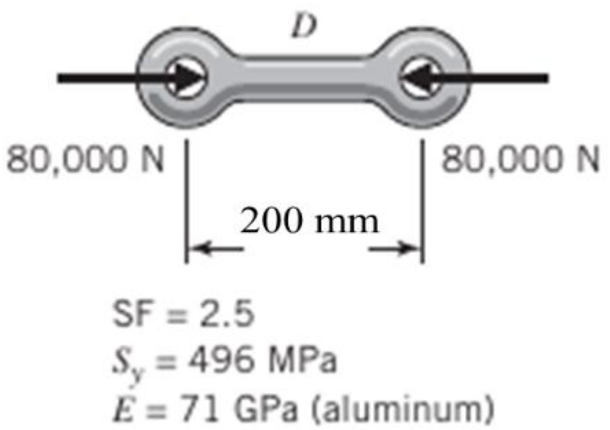
Check the slenderness ratio: $(l/k) = 106$

Check $(l/k)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2} = \left(\frac{2\pi^2 209(10^9)}{689(10^6)}\right)^{1/2} = 77$

Therefore Euler's equation is applicable (assumption is ok)

Example 1 - extension

- Repeat example 1, except reduce the length to 200 mm and use aluminum with properties of S_y = 496MPa, E = 71GPa.



Example 1 - extension

Assume Euler's equation: $\frac{P_{cr}}{A} = \frac{C\pi^2 E}{(l/k)^2}$

$$\frac{200(10^3)}{\pi r^2} = \frac{\pi^2(71)(10^9)}{(0.2 \times 2/r)^2}$$

Radius *r* =0.011m or diameter *d* = 0.022m

Check the slenderness ratio: $(l/k) = 36.4$

Check $(l/k)_1 = \left(\frac{2\pi^2 CE}{S_y}\right)^{1/2} = \left(\frac{2\pi^2 71(10^9)}{496(10^6)}\right)^{1/2} = 53$

Assumption wrong. Apply parabolic equation: $\frac{P_{cr}}{A} = S_y - \left(\frac{S_y}{2\pi}\frac{l}{k}\right)^2 \frac{1}{CE}$

$$\frac{200(10^3)}{\pi r^2} = 496(10^6) - \left(\frac{496(10^6)}{2\pi}\frac{0.2}{r/2}\right)^2 \frac{1}{71(10^9)}$$

Radius *r* =0.0125m or diameter *d* = 0.025m

Slenderness ratio $(l/k) = 32$

Columns with eccentric loading

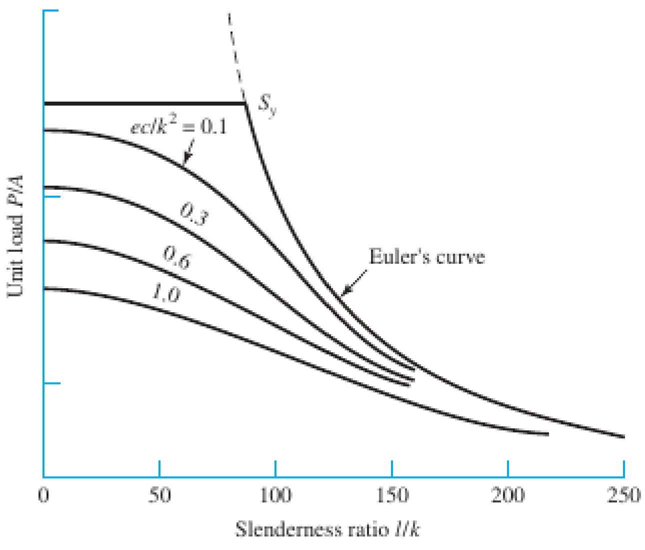
An eccentric load is one that applied away from the centroidal axis of the Cross section of column

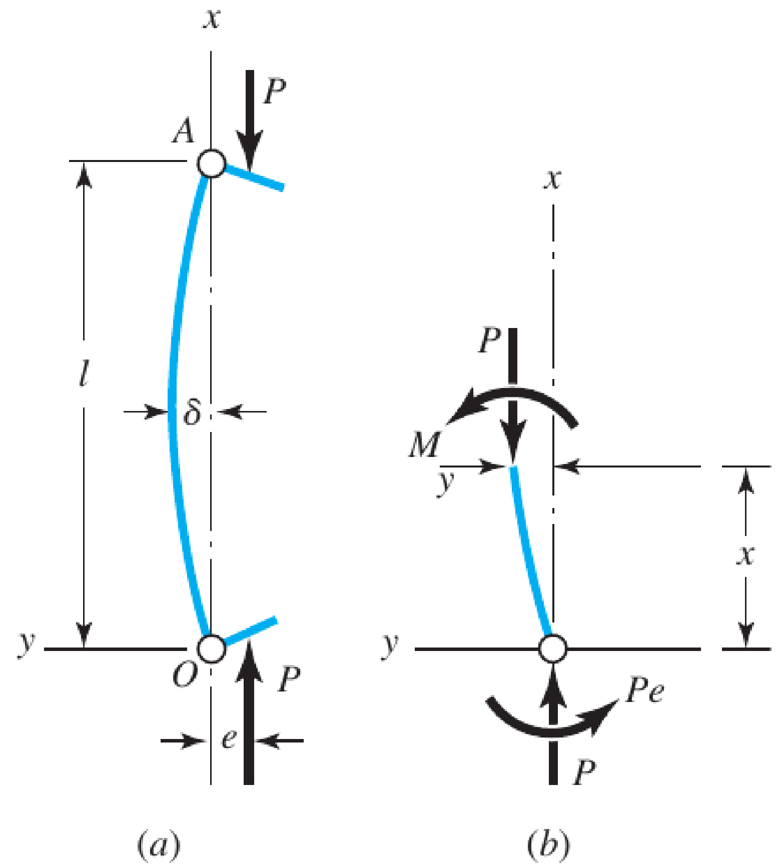
Columns with eccentric loading

Use the secant column formula:

$$\sigma_c = \frac{P}{A}\left\{1 + (ec/k^2)\sec\left[(l/2k)\sqrt{P/AE}\right]\right\}$$

Note: (ec/k^2) = eccentricity ratio



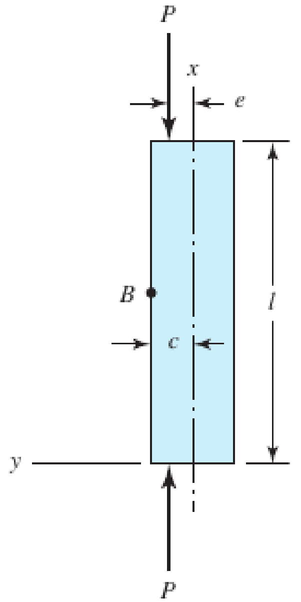


Note: maximum stress and deflection occur in the outermost fibers of the cross section at the mid-length of the column

Short compression members

The magnitude of the maximum compressive stress in the *x* direction at point *B* in a strut or short compression member is the sum of a simple component (*P*/*A*) and a flexural component (*Mc*/*I*); i.e.

$$\sigma_c = \frac{P}{A}\left(1 + \frac{ec}{k^2}\right)$$



Note: (k) = radius of gyration, *e* = eccentricity of loading, and *c* = distance of point "*B*" from neutral axis

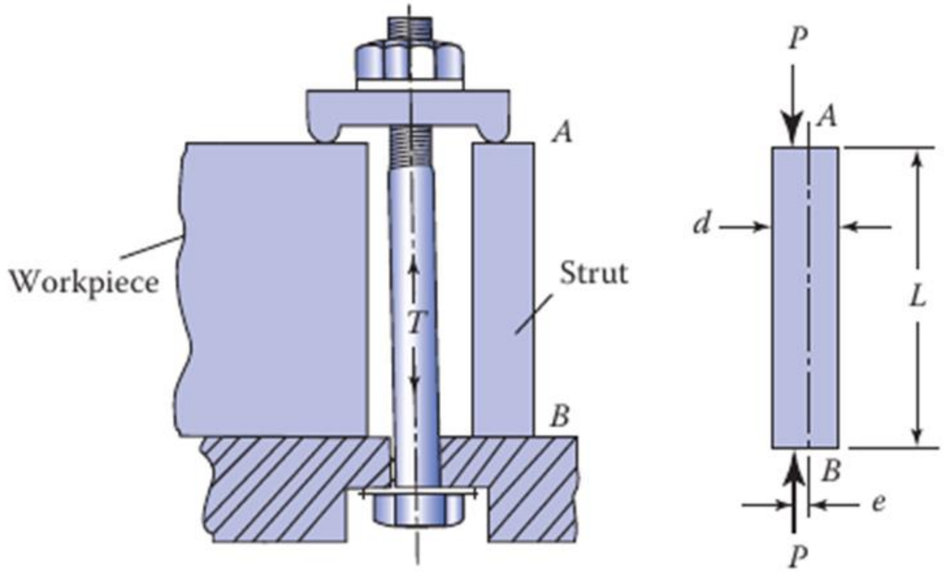
- To determine if the member is short, determine

$$\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2}$$

- Apply column with eccentric loading equation if slenderness ratio is greater than $(l/k)_2$ and assume short compression member otherwise

Example 2

A piece of work in the process of manufacture is attached to a cutting machine table by a bolt tightened to a tension of *T*. The clamp contact is offset from a centroidal axis of the strut *AB* by a distance "*e*" as shown. The strut is made of a structural ASTM 36 steel of diameter *d* and length *L*. Given: The numerical values are d = 30 mm, e = 3.5 mm, L = 125 mm, $T = 2P$ = 7 kN, E = 200 GPa, Find the largest stress in the strut. Assume strut is pinned at both ends.



Example 2

The cross-sectional area properties of the strut are:

Radius r = 15mm;

Area $A = \pi r^2$ = 706.9 mm^2;

Area moment of inertia $I = (1/4)\pi r^4$ = 39.76 mm^4;

Radius of gyration $k = \sqrt{(I/A)}$ = 7.5 mm;

Slenderness ratio: $(l/k) = 16.67$ and $\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2} = 56.7$

Short column is applicable:

$$\sigma_c = \frac{P}{A}\left[1 + \frac{ec}{k^2}\right] = \frac{3500}{706.9(10^{-6})}\left[1 + \frac{3.5(10^{-3})15(10^{-3})}{(7.5\times10^{-3})^2}\right] = 9572\text{kPa}$$

Example 2 – extension

Repeat example 2 using the secant formula

The cross-sectional area properties of the strut are:

Radius r = 15mm;

Area $A = \pi r^2 = 706.9$ mm^2;

Area moment of inertia $I = (1/4)\pi r^4 = 39.76$ mm^4;

Radius of gyration $k = \sqrt{(I/A)} = 7.5$ mm;

Slenderness ratio: $(l/k) = 16.67$ and $\left(\frac{l}{k}\right)_2 = 0.282\left(\frac{AE}{P}\right)^{1/2} = 56.7$

Secant formula:

$$\sigma_c = \frac{P}{A}\left\{1 + (ec/k^2)\sec\left[(l/2k)\sqrt{P/AE}\right]\right\} = 9572\text{kPa}$$

The results indicate that for this short column, the effect of the lateral deflection on stress can be omitted

Shock & impact

- Impact refers to the collision of two masses with initial relative velocity
- Shock is a more general term that is used to describe any suddenly applied force or disturbance. Thus the study of shock includes impact as a special case

Consider the case of a freely falling weight $W = mg$

The change in PE of the mass = $mgh + mg\delta = Wh + W\delta$

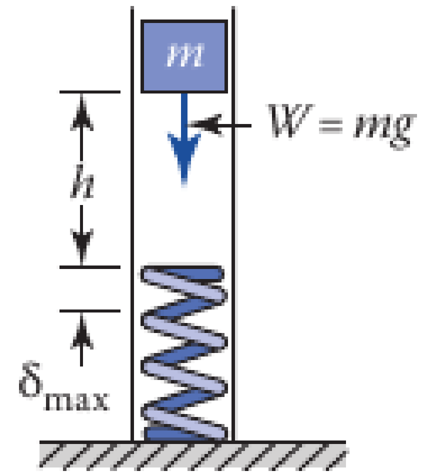
This energy is stored in the spring = $\frac{1}{2}k\delta^2$

Hence $\frac{1}{2}k\delta^2 = Wh + W\delta$ or $\delta^2 - \frac{2W}{k}\delta - \frac{2Wh}{k} = 0$

The roots of the quadratic equation are

$$\delta = \frac{W}{k} \pm \frac{1}{2}\sqrt{\left(\frac{2W}{k}\right)^2 + 4\left(\frac{2Wh}{k}\right)} = \frac{W}{k} \pm \frac{W}{k}\sqrt{1 + \left(\frac{2hk}{W}\right)}$$

Force $F = k\delta = W \pm W\sqrt{1 + \left(\frac{2hk}{W}\right)}$



Shock & impact

Note that we can define static deflection as $\delta_{st} = \frac{W}{k}$

When $h >> \delta_{st}$ the $W\delta$ can be neglected and $\delta_{max} = \sqrt{2\delta_{st}h}$

Otherwise $\delta = \frac{W}{k} \pm \frac{W}{k}\sqrt{1 + \left(\frac{2hk}{W}\right)} = \delta_{st} \pm \delta_{st}\sqrt{1 + \left(\frac{2h}{\delta_{st}}\right)}$

and $\delta_{max} = \delta_{st}\left(1 + \sqrt{1 + \left(\frac{2h}{\delta_{st}}\right)}\right)$

Define impact factor as $\delta_{max} = K\delta_{st}$ where $K = 1 + \sqrt{1 + \left(\frac{2h}{\delta_{st}}\right)}$

Maximum stress due to impact $\sigma_{max} = K\sigma_{st}$

Force $F = k\delta = W \pm W\sqrt{1 + \left(\frac{2hk}{W}\right)}$

When h =0 maximum force $F_{max} = 2W$

This says that when the weight is released while in contact with the spring but is not exerting any force on the spring, the largest force is double the weight

Example 3

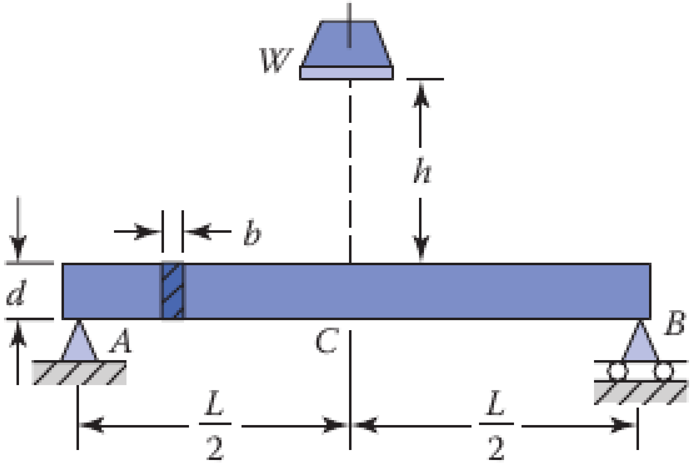
A weight W is dropped from a height *h*, striking at midspan a simply supported steel beam of length *L*. The beam is of rectangular cross section of width *b* and depth *d*. Calculate the maximum deflection and maximum stress for these two cases:

a) The beam is rigidly supported at each end.

b) The beam is supported at each end by springs.

Given: W = 100 N, h = 150 mm, L = 2 m, b = 30 mm, and d = 60 mm

Modulus of elasticity E = 200 GPa and spring rate k = 200 kN/m.



Example 3

The maximum deflection, due to a static load, is $\delta_{st} = \frac{WL^3}{48EI} = 0.154$mm

The maximum moment (occurs at C) is $M=WL/4$

Maximum static stress equals $\sigma_{st} = \frac{Mc}{I}$ =2.778MPa

Impact factor as $K = 1 + \sqrt{1 + \left(\frac{2h}{\delta_{st}}\right)}$ = 45.15

$\delta_{max} = K\delta_{st}$= 6.95 mm

$\sigma_{max} = K\sigma_{st}$= 125 MPa

If beam is supported by springs, the static deflection of the beam due to its own bending and the deformation of the springs at the ends is

$\delta_{st} = \frac{W/2}{k} + \frac{WL^3}{48EI} = \frac{50}{200} + 0.154$ = 0.404mm

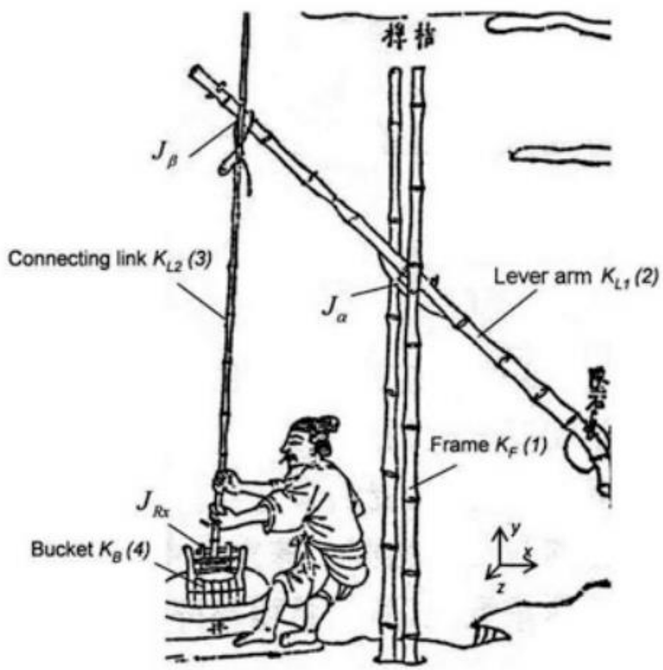
New impact factor as $K = 1 + \sqrt{1 + \left(\frac{2h}{\delta_{st}}\right)}$ = 28.27

$\delta_{max} = K\delta_{st}$= 11.42 mm

$\sigma_{max} = K\sigma_{st}$= 78.53 MPa

Ancient Chinese mechanisms

Jie Gao (桔 槔) for water lifting



How would you analyse the 2 supporting bamboo columns?