# MEMS1028
# Mechanical Design 1

Lecture 6

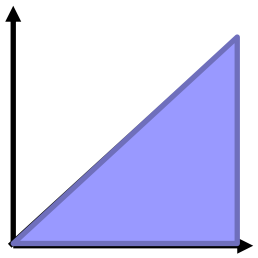
Advanced deformation analysis (Castigliano’s theorem)

# Objectives

- Explain the concept of strain energy and determine strain energies for common loadings
- Apply Castigliano’s theorem to analyze the deflections of common engineering elements
- Solve statically indeterminate structures common encountered in engineering applications

# Spring potential energy

- The work done in deforming the spring is stored as potential energy
- The energy is recovered when the load is removed
- The energy is the area under the force-displacement curve
- For a linear force-displacement, the potential energy is given by




$$U = \frac{1}{2}Fy = \frac{1}{2}ky^2 = \frac{1}{2}F\frac{F}{k} = \frac{F^2}{2k}$$

- This potential energy causes the deformation or strain and is also know as the strain energy

# Strain energy

Applied loading deforms an elastic material which results in strain. In the process, external work done on the elastic member is transformed into strain energy (this is a type of potential energy which is recoverable in the elastic zone when the loading is removed)

The strain energy can be written in terms of the spring rate as $U = \frac{F^2}{2k}$

For tension and compression: $k = \frac{F}{\delta} = \frac{AE}{L}$ and $U = \frac{F^2 L}{2AE}$ for constant area or in general form as $U = \int \frac{F^2}{2AE} dx$

For torsion of circular bar: $k = \frac{T}{\phi} = \frac{GJ}{L}$ and $U = \frac{T^2 L}{2GJ}$ for constant polar moment of inertia; or in general form as $U = \int \frac{T^2}{2GJ} dx$

# Strain energy

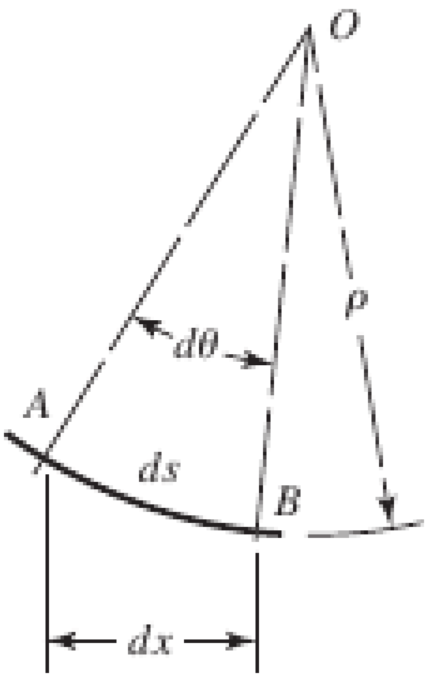
The bending $dU = \frac{1}{2} M d\theta$

where $ds = \rho d\theta$ and $\rho = \frac{EI}{M}$

Hence $dU = \frac{1}{2} M d\theta = \frac{M^2 ds}{2EI}$

Or in general form as $U = \int \frac{M^2}{2EI} ds$

For constant $I$: $U = \frac{M^2 L}{2EI}$



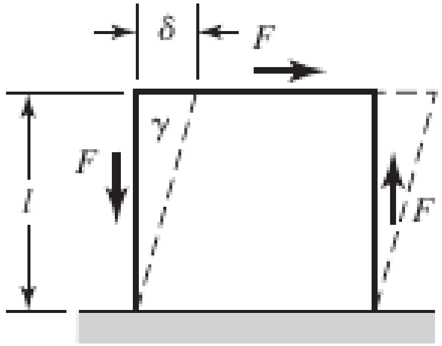
# Strain energy

The direct shear $U = \frac{1}{2}F\delta$

where shear strain is $\gamma = \frac{\delta}{L} = \frac{\tau}{G} = \frac{F}{AG}$

Hence $U = \frac{1}{2}F\delta = \frac{F^2L}{2AG}$ for constant area or

In general form as $U = \int \frac{F^2}{2AG}dx$



# Strain energy

For transverse shear in bending:

$$U = \int \frac{CV^2}{2AG} dx$$

For constant area:

$$U = \frac{CV^2 L}{2AG}$$

$C$ = correction factor:

**Table 4-1**

Strain-Energy Correction Factors for Transverse Shear

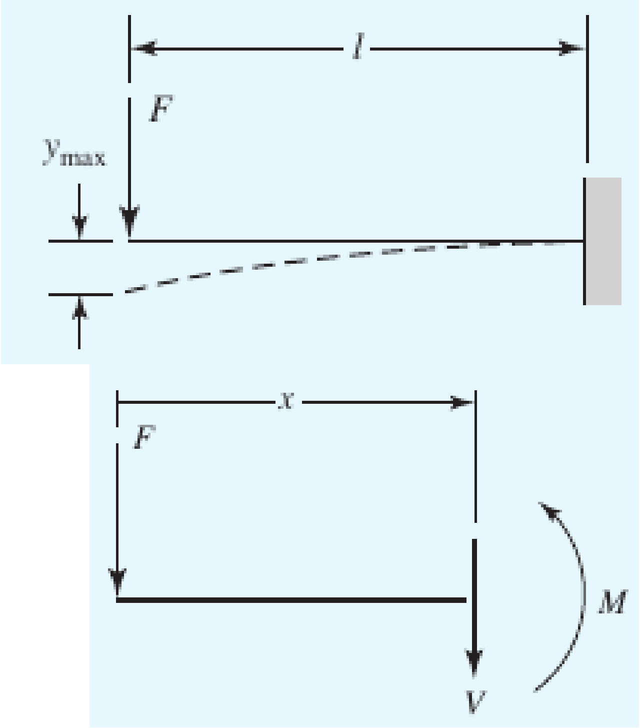
*Source:* Richard G. Budynas, *Advanced Strength and Applied Stress Analysis,* 2nd ed., McGraw-Hill, New York, 1999. Copyright © 1999 The McGraw-Hill Companies.

| Beam Cross-Sectional Shape | Factor C |
|---|---|
| Rectangular | 1.2 |
| Circular | 1.11 |
| Thin-walled tubular, round | 2.00 |
| Box sections† | 1.00 |
| Structural sections† | 1.00 |

†Use area of web only.

# Illustrative example 1

A cantilever beam with a round cross section has a concentrated load *F* at the end, as shown. Find the strain energy in the beam




The FBD is shown for $0 \le x \le L$ and the shear and bending moment are

$$M = -Fx$$
$$V = -F$$

Both bending moment and shear contributed to the strain energy:

$$U = \int \frac{M^2}{2EI} dx + \int \frac{CV^2}{2AG} dx = \int \frac{F^2 x^2}{2EI} dx + \int \frac{CF^2}{2AG} dx$$

For round cross section $C$ =1.11 and

$$U = \frac{F^2 L^3}{6EI} + \frac{1.11 F^2 L}{2AG}$$

# Castigliano's Theorem

When a body is elastically deflected by any combination of loads, the deflection at any point and in any direction is equal to the partial derivative of strain energy (computed with all loads acting) with respect to a load located at that point and acting in that direction, i.e.

$$\delta_i = \frac{\partial U}{\partial F_i} \text{ or } \theta_i = \frac{\partial U}{\partial M_i} \text{ or } \phi_i = \frac{\partial U}{\partial T_i}$$

Note:

$\delta_i$ is the linear displacement at point "$i$" (tension/compression or bending);
$F_i$ is the force applied at point "$i$" (tension/compression or bending);
$\theta_i$ is the angular displacement at point "$i$" (bending);
$M_i$ is the moment applied at point "$i$" (bending);
$\phi_i$ is the angular twist at point "$i$" (torsion);
$T_i$ is the torque applied at point "$i$" (torsion);
$U$ is the total strain energy computed with all loads acting

# Castigliano's Theorem

- In applying Castigliano's theorem, we must find the total strain energy of all external loadings
- Typically, the total strain energy can be related to the loads through expression like

$$U = \int_0^L \frac{F^2 dx}{2AE} + \int_0^L \frac{T^2 dx}{2GJ} + \int_0^L \frac{M^2 dx}{2EI} + \int_0^L \frac{CV^2 dx}{2AG}$$

- Note that $\delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{F\frac{\partial F}{\partial F_i}dx}{AE} + \int_0^L \frac{T\frac{\partial T}{\partial F_i}dx}{GJ} + \int_0^L \frac{M\frac{\partial M}{\partial F_i}dx}{EI} + \int_0^L \frac{CV\frac{\partial V}{\partial F_i}dx}{AG}$

- If a system consists of several members, the above will have to include summations over all members in the system

# Castigliano's Theorem

$$\delta_i = \frac{\partial U}{\partial F_i} \text{ and } \theta_i = \frac{\partial U}{\partial M_i}$$

For axially loaded bar (under single force) in tension or compression:

$$U = \int \frac{F^2}{2AE} dx \text{ and } \delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{F\frac{\partial F}{\partial F_i}dx}{AE} = \frac{FL}{AE}$$

For torsion of circular bar (under single torque):

$$U = \int \frac{T^2}{2GJ} dx \text{ and } \phi_i = \frac{\partial U}{\partial T_i} = \int_0^L \frac{T\frac{\partial T}{\partial T_i}dx}{GJ} = \frac{TL}{GJ}$$

# Castigliano's Theorem

- In the case of a beam, to find the linear and angular deflections in bending:
  - Linear displacement at "*i*" is expressed as $\delta_i = \frac{\partial U}{\partial F_i} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F_i}dx;$
  - Angular displacement at "*i*" is expressed as $\theta_i = \frac{\partial U}{\partial M_i} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_i}dx;$
- When it is necessary to obtain the displacement at a point where there is no corresponding load, we must first place a fictitious load at that point in the direction of the desired displacement. The displacement is then obtained by first differentiating the strain energy with respect to the fictitious load and then setting the fictitious load equal to zero

# Castigliano's Theorem

- In the case of a truss consisting of "$n$" members of lengths "$L_j$", axial rigidity "$A_jE_j$" and internal force "$F_j$", the strain energy can be found by

$$U = \sum_{j=1}^{n} \frac{F_j^2 L_j}{2A_j E_j}$$

  - Linear displacement at point "$i$" is expressed as

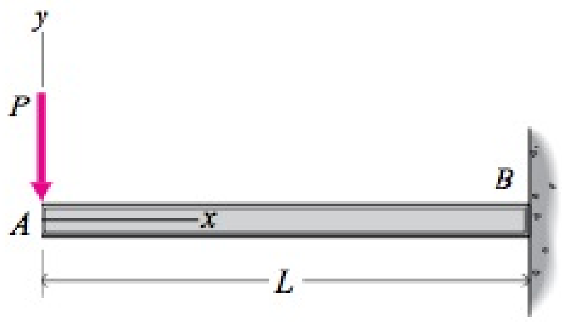
$$\delta_i = \frac{\partial U}{\partial F_i} = \sum_{j=1}^{n} \frac{F_j L_j}{A_j E_j} \frac{\partial F_j}{\partial F_i};$$

- Similar to bending, it is necessary to apply a fictitious load to obtain the displacement at a point where there is no corresponding load

# Summary table

| Load type | General strain energy | General deflection |
| --- | --- | --- |
| Axial | $U = \int_0^L \frac{F^2}{2AE} dx$ | $\delta_i = \int_0^L \frac{F \frac{\partial F}{\partial F_i} dx}{AE}$ |
| Bending | $U = \int_0^L \frac{M^2 dx}{2EI}$ | $\delta_i = \int_0^L \frac{M \frac{\partial M}{\partial F_i} dx}{EI}$; $\theta_i = \int_0^L \frac{M \frac{\partial M}{\partial M_i} dx}{EI}$ |
| Transverse shear | $U = \int_0^L \frac{CV^2 dx}{2AG}$ | $\delta_i = \int_0^L \frac{CV \frac{\partial V}{\partial F_i} dx}{AG}$ |
| Torsion | $U = \int_0^L \frac{T^2 dx}{2GJ}$ | $\phi_i = \int_0^L \frac{T \frac{\partial T}{\partial F_i} dx}{GJ}$ |

# Example 1




The cantilever beam shown is subjected to a concentrated load $P$ at the left end. Determine the deflection and slope at point $A$ using Castigliano's theorem (assume negligible strain energy due to transverse shear)

The total strain energy is due only to the bending moment: $U = \int \frac{M^2}{2EI} dx$

Note that $\delta_i = \frac{\partial U}{\partial F_i}$

There is a force $P$ at "$A$" to find the linear displacement at "$A$"

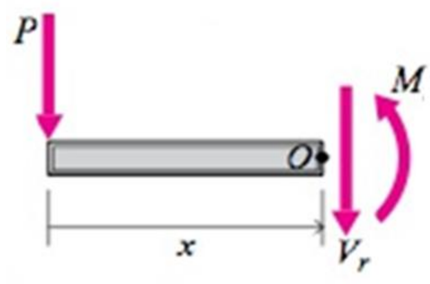
The bending moment for $0 \le x \le L$ is found to be:

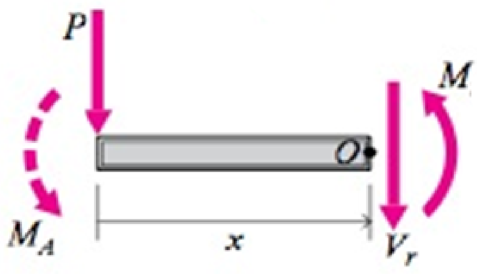
$$M = -Px$$

$\frac{\partial M}{\partial P} = -x$ and $\delta_A = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P} dx = \int_0^L \frac{(Px)x}{EI} dx = \int_0^L \frac{Px^2}{EI} dx = \frac{PL^3}{3EI}$

Deflection at point $A$ *is* $\delta_A = \frac{PL^3}{3EI}$

# Example 1




To find the angular displacement at *A*, there must be a bending moment at *A*

There is no bending moment at "*A*" to find the angular displacement at "*A*"

A fictitious bending moment $M_A$ has to be added at point "*A*"

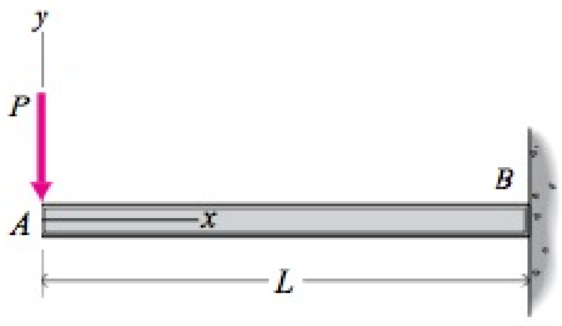
The bending moment for $0 \le x \le L$ is now

$$M = -Px - M_A$$

$\frac{\partial M}{\partial M_A} = -1$ and $\theta_A = \frac{\partial U}{\partial M_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_A}dx = \int_0^L \frac{(Px - M_A)}{EI}dx = \int_0^L \frac{Px}{EI}dx = \frac{PL^2}{2EI}$

Slope or angular displacement at point *A* is $\theta_A = \frac{PL^2}{2EI}$

# Example 1 – extension




Determine the deflection at point *A* in example 1 using Castigliano's theorem (if the strain energy due to shear is not negligible and the beam has a rectangular cross section)

The total strain energy is due only to the bending moment and shear:

$$U = \int \frac{M^2}{2EI} dx + \int \frac{CV^2}{2AG} dx$$

Note that $\delta_i = \frac{\partial U}{\partial F_i}$ and for rectangular cross section $C = 1.2$

There is a force *P* at "*A*" to find the linear displacement at "*A*"

The bending moment and shear force for $0 \le x \le L$ is

$$M = -Px \text{ and } V = -P$$

$$\frac{\partial M}{\partial P} = -x$$

# Example 1 – extension

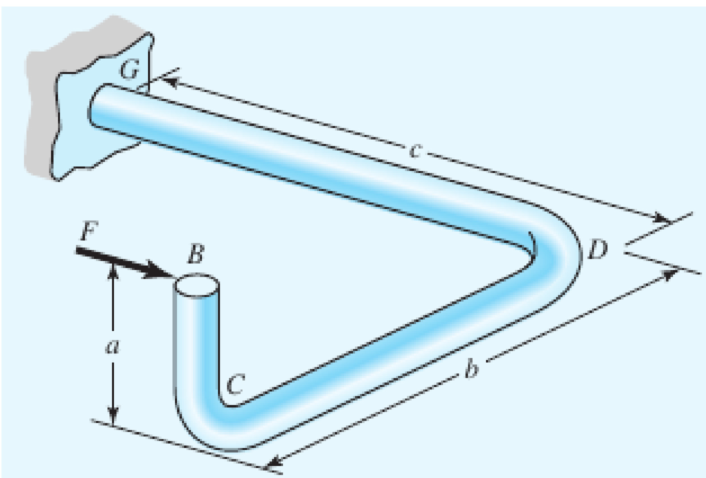
$$\delta_A = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dx + \int_0^L \frac{CV}{AG}\frac{\partial V}{\partial P}dx = \int_0^L \frac{(Px)x}{EI}dx + \int_0^L \frac{CP}{AG}dx$$

$$\delta_A = \int_0^L \frac{Px^2}{EI}dx + \int_0^L \frac{CP}{AG}dx = \frac{PL^3}{3EI} + \frac{CPL}{AG}$$

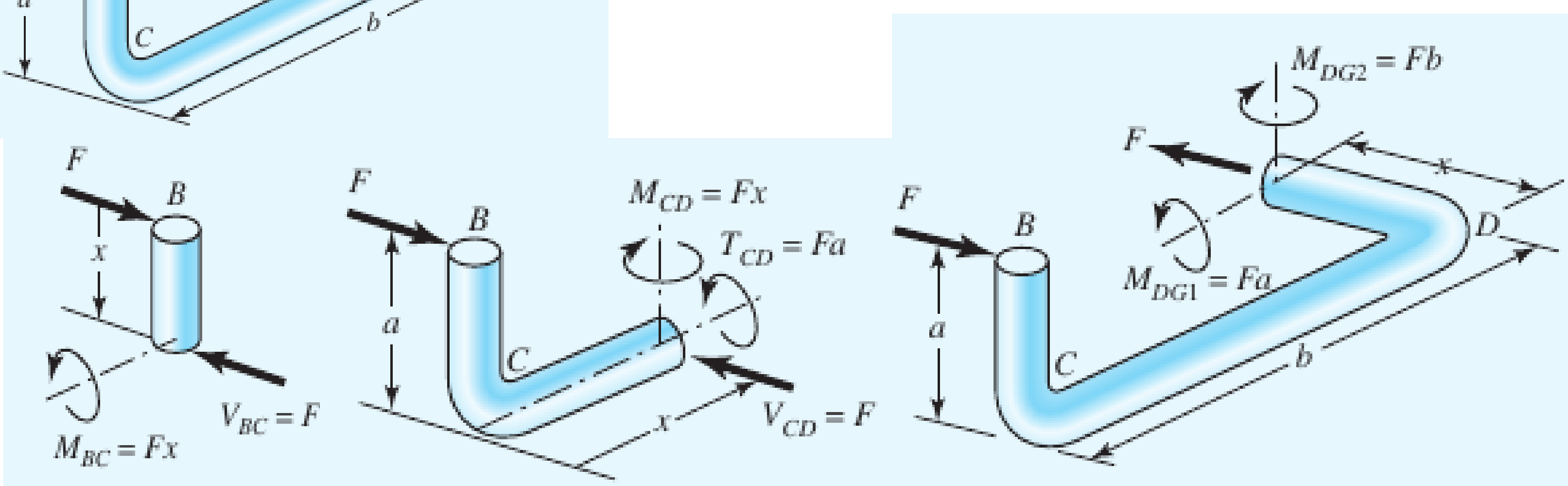
Deflection at point $A$ *is* $\delta_A = \frac{PL^3}{3EI} + \frac{1.2PL}{AG}$

# Example 2

For the wire form of diameter *d* shown, determine the deflection of point *B* in the direction of the applied force *F* (neglect the effect of transverse shear)




Cut the wire at different sections and draw the FBDs. Use the FBDs to determine the loadings in the various sections



# Example 2

Along BC, the element is in bending and $M = Fx$ for $(0 \le x \le a)$

$$\delta_{BC} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx = \int_0^L \frac{Fx^2}{EI}dx = \frac{Fa^3}{3EI}$$

Along CD, the element is in bending and torsion;
For bending $M = Fx$ for $(0 \le x \le b)$

$$\delta_{CD\_bend} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx = \int_0^L \frac{Fx^2}{EI}dx = \frac{Fb^3}{3EI}$$

For torsion $T = Fa$ for $(0 \le x \le b)$

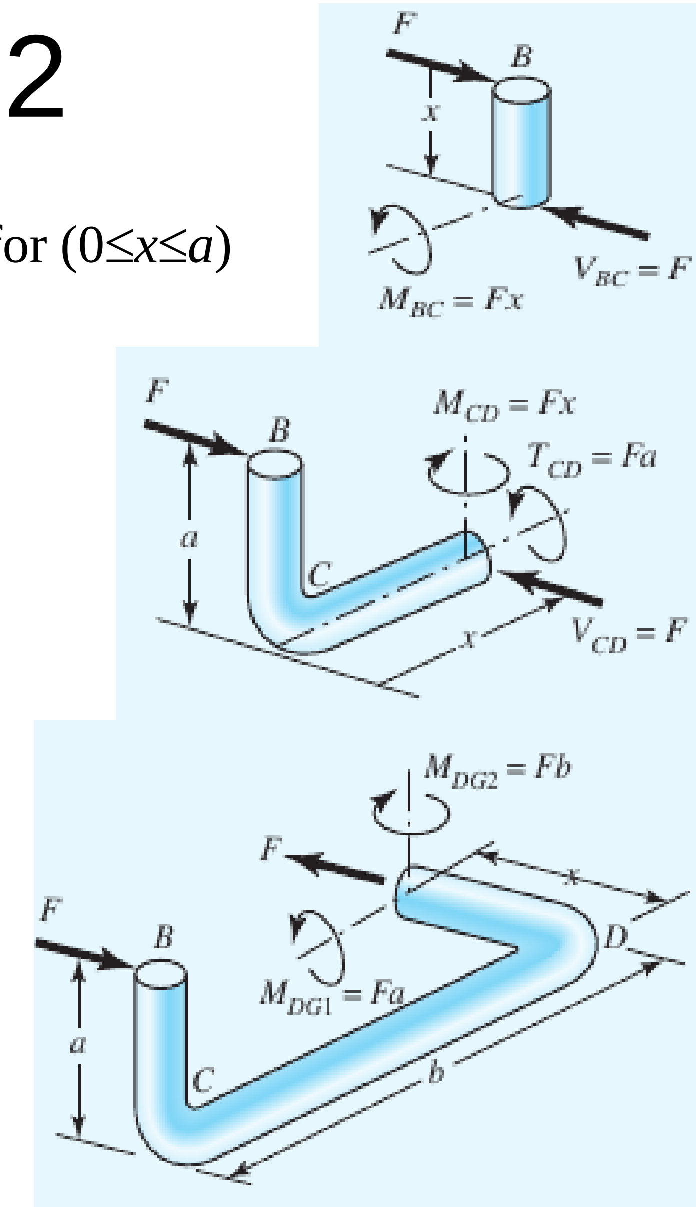
$$\delta_{CD\_tor} = \int_0^L \frac{T}{GJ}\frac{\partial T}{\partial F}dx = \int_0^L \frac{Fa^2}{GJ}dx = \frac{Fa^2b}{GJ}$$

Along DG, the element is in bending in 2 planes;
For 1st bending $M = Fa$ for $(0 \le x \le c)$

$$\delta_{DG\_bend1} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx = \int_0^L \frac{Fa^2}{EI}dx = \frac{Fa^2c}{EI}$$

For 2nd bending $M = Fb$ for $(0 \le x \le c)$

$$\delta_{DG\_bend2} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}dx = \int_0^L \frac{Fb^2}{EI}dx = \frac{Fb^2c}{EI}$$

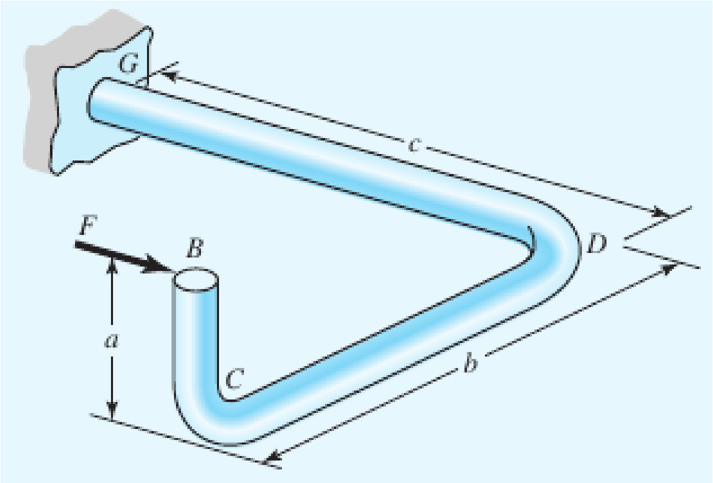


# Example 2

The total displacement of the wire is

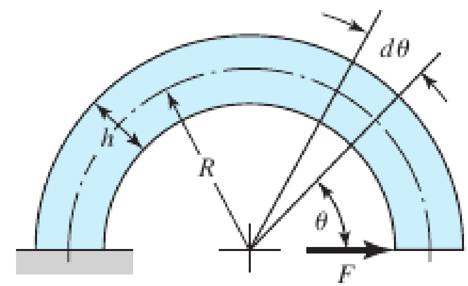
$$\delta = \delta_{BC} + \delta_{CD\_bend} + \delta_{CD\_tor} + \delta_{DG\_bend1} + \delta_{DG\_bend2}$$

$$\delta = \frac{Fa^3}{3EI} + \frac{Fb^3}{3EI} + \frac{Fa^2b}{GJ} + \frac{Fa^2c}{EI} + \frac{Fb^2c}{EI}$$



# Deflection of curved beams

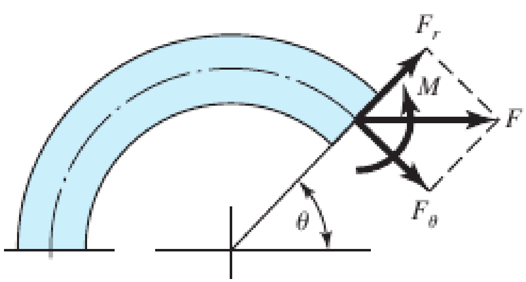
- A curve beam with an applied force $F$ is shown
- A section at angle $\theta$ will have 3 internal loading: a moment $M$; a shear force $F_r$ and a normal force $F_\theta$
- The total strain energy due to the internal loads are

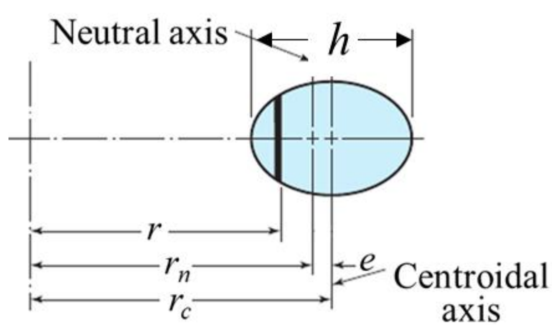



$$U = \int \frac{M^2 d\theta}{2AeE} + \int \frac{F_\theta^2 R d\theta}{2AE} - \int \frac{M F_\theta d\theta}{AE} + \int \frac{C F_r^2 R d\theta}{2AG}$$

The deflection produced by force $F$ is given by

$$\delta = \frac{\partial U}{\partial F} = \int \frac{M\left(\frac{\partial M}{\partial F}\right) d\theta}{AeE} + \int \frac{F_\theta R\left(\frac{\partial F_\theta}{\partial F}\right) d\theta}{AE} - \int \frac{\left(\frac{\partial M F_\theta}{\partial F}\right) d\theta}{AE} + \int \frac{C F_r R\left(\frac{\partial F_r}{\partial F}\right) d\theta}{AG}$$




| Beam Cross-Sectional Shape | Factor C |
|---|---|
| Rectangular | 1.2 |
| Circular | 1.11 |
| Thin-walled tubular, round | 2.00 |
| Box sections[†] | 1.00 |
| Structural sections[†] | 1.00 |

# Deflection of curved beams

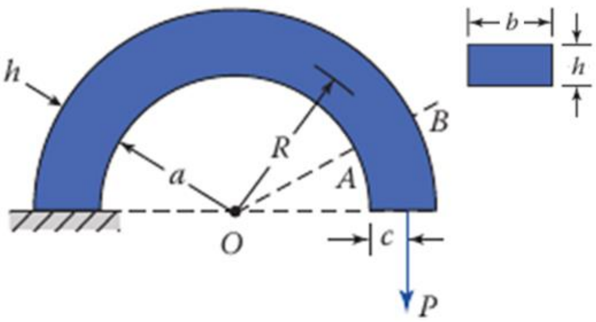
- If the radius is significantly greater than the thickness, the eccentricity "*e*" can be ignored
- Generally, the approximated deflection can be simplified by substituting

$dx = Rd\theta$ in $U = \int_0^L \frac{F^2 dx}{2AE} + \int_0^L \frac{M^2 dx}{2EI} + \int_0^L \frac{CV^2 dx}{2AG}$ which leads to

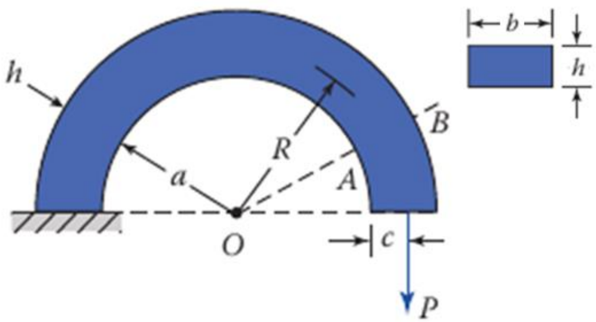
$$\delta = \frac{\partial U}{\partial F} = \int_0^{\theta} \frac{F \frac{\partial F}{\partial F_i} Rd\theta}{AE} + \int_0^{\theta} \frac{M \frac{\partial M}{\partial F_i} Rd\theta}{EI} + \int_0^{\theta} \frac{CV \frac{\partial V}{\partial F_i} Rd\theta}{AG}$$

# Example 3

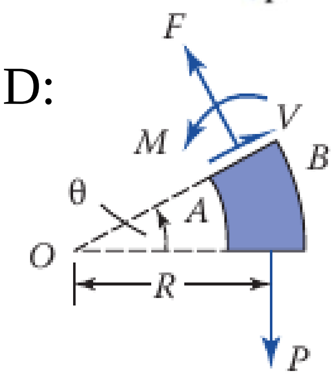
A load of *P* is applied to a steel curved frame, as shown. Develop an expression for the vertical deflection δ of the free end by considering the effects of the internal normal and shear forces in addition to the bending moment. Calculate the value of δ for the following data:
$a$ = 60 mm, $P$ = 10kN, $h$ = 30 mm, $b$ = 15 mm, $E$ = 210GPa, $G$ = 80GPa



# Example 3




FBD:




- Note that

$$F = P\cos\theta$$
$$V = P\sin\theta$$

- Taking moments about O

$$M - PR + FR = 0$$
$$M = PR - PR\cos\theta$$
$$M = PR(1 - \cos\theta)$$

$$\frac{\partial F}{\partial P} = \cos\theta, \frac{\partial V}{\partial P} = \sin\theta, \text{and } \frac{\partial M}{\partial P} = R(1 - \cos\theta)$$

# Example 3

$$\delta = \frac{\partial U}{\partial F} = \int_0^\theta \frac{F\frac{\partial F}{\partial P}Rd\theta}{AE} + \int_0^\theta \frac{M\frac{\partial M}{\partial P}Rd\theta}{EI} + \int_0^\theta \frac{CV\frac{\partial V}{\partial P}Rd\theta}{AG}$$ where C=1.2 (rectangular)

$$\delta = \int_0^\pi \frac{PR(\cos\theta)^2\,d\theta}{AE} + \int_0^\pi \frac{PR^3(1-\cos\theta)^2 d\theta}{EI} + \int_0^\pi \frac{CPR\,(\sin\theta)^2 d\theta}{AG}$$

$$\delta_1 = \int_0^\pi \frac{PR(\cos\theta)^2 d\theta}{AE} = \int_0^\pi \frac{PR(1+\cos 2\theta)d\theta}{2AE} = \left[\frac{PR(\theta+\frac{1}{2}\sin 2\theta)}{2AE}\right]_0^\pi = \frac{PR\pi}{2AE}$$

$$\delta_2 = \int_0^\pi \frac{PR^3(1-\cos\theta)^2 d\theta}{EI} = \int_0^\pi \frac{PR^3}{EI}\left[\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right]$$

$$= \frac{PR^3}{EI}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi = \frac{3PR^3\pi}{2EI}$$

# Example 3

$$\delta = \int_0^{\pi} \frac{PR(\cos\theta)^2\, d\theta}{AE} + \int_0^{\pi} \frac{PR^3(1-\cos\theta)^2 d\theta}{EI} + \int_0^{\pi} \frac{CPR\,(\sin\theta)^2 d\theta}{AG}$$

$$\delta_3 = \int_0^{\pi} \frac{CPR\,(\sin\theta)^2 d\theta}{AG} = \int_0^{\pi} \frac{CPR(1-\cos 2\theta)d\theta}{2AG} = \left[\frac{CPR(\theta - \frac{1}{2}\sin 2\theta)}{2AG}\right]_0^{\pi} = \frac{CPR\pi}{2AG}$$

$$\delta = \delta_1 + \delta_2 + \delta_3 = \frac{PR\pi}{2AE} + \frac{3PR^3\pi}{2EI} + \frac{CPR\pi}{2AG}$$

$a$ = 60 mm, $P$ = 10kN, $h$ = 30 mm, $b$ = 15 mm, $E$ = 210GPa, $G$ = 80GPa

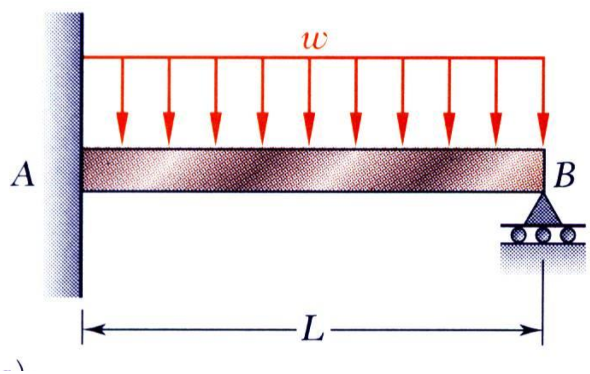
Area $A = bh = 4.5 \times 10^{-4}$m$^2$; $I = \frac{bh^3}{12} = 337.5 \times 10^{-10}$m$^4$;

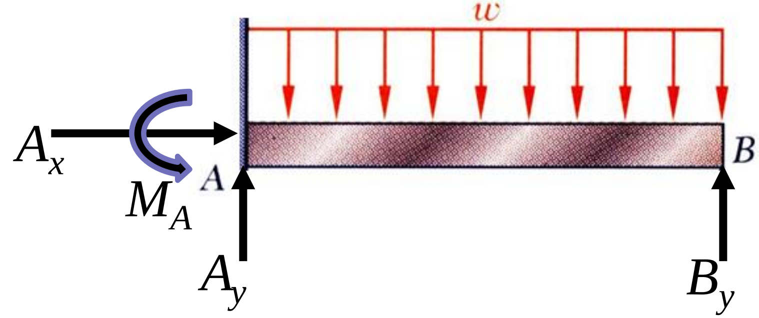
For rectangular cross section $C = 1.2$; $R = a + 0.5h = 0.075$m;

$$\delta = 0.01(10^{-3}) + 2.81(10^{-3}) + 0.04(10^{-3}) = 2.86(10^{-3})\text{m}$$

# Statically indeterminate problems

- A system is over-constrained when the static equilibrium equations (i.e. force and moment equilibrium conditions) are insufficient to determine the internal forces and reactions
- Such a system is said to be statically indeterminate as it has more unknown support (reaction) forces and/or moments than static equilibrium equations
- The extra constraint supports are called redundant supports
- Example:



# Statically indeterminate problems

Procedure for analyzing statically indeterminate problems:

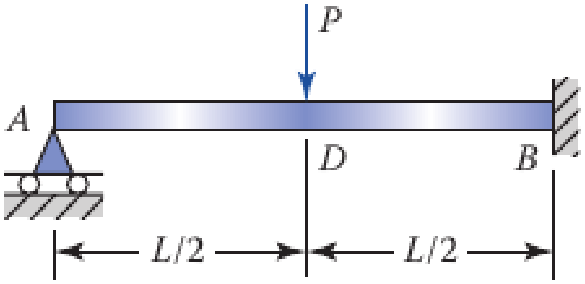
1) Choose the redundant reaction "*R*"
2) Write the equations of static equilibrium for the remaining reactions in terms of the applied loads and "*R*" of step 1
3) All external forces, including both loads and redundant reactions, must generate displacements compatible with the original supports. Write the deflection equation(s) for the point at the location of the redundant reaction of step 1 in terms of the applied loads and "*R*". Normally the deflection is zero and we can determine "*R*" by solving

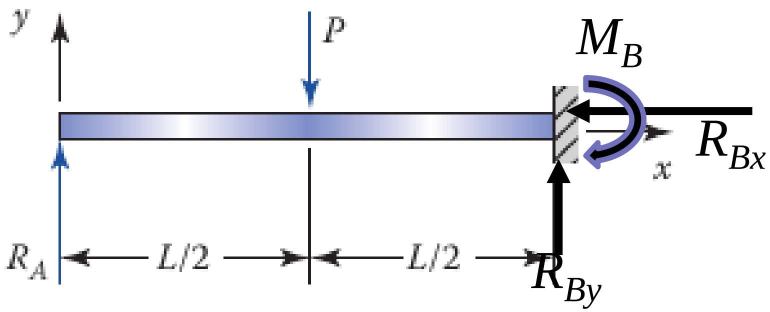
$$\delta = \frac{\partial U}{\partial R} = 0$$

4) The equations from steps 2 and 3 can now be solved to determine the reactions (Note: If a redundant reaction is a moment, the corresponding deflection equation is a rotational deflection)

# Example 4

A propped cantilevered beam carries a concentrated load $P$ at its midspan. Find the support reactions




1) Choose the redundant reaction "$R_A$"
2) Write the equations of static equilibrium

$$R_{Bx} = 0;$$
$$R_{By} = P - R_A;$$
$$M_B = P\frac{L}{2} - R_A L$$

3) At "$A$" the deflection is zero:

$$\delta = 0 = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_A}dx$$

For $0 \le x \le L/2$: $M = R_A x;\ \frac{\partial M}{\partial R_A} = x$

For $L/2 \le x \le L$: $M = R_A x - P\left(x - \frac{L}{2}\right);\ \frac{\partial M}{\partial R_A} = x$

# Example 4

Combining the terms:

For $0 \le x \le L/2$: $M = R_A x; \frac{\partial M}{\partial R_A} = x$

For $L/2 \le x \le L$: $M = R_A x - P(x - \frac{L}{2}); \frac{\partial M}{\partial R_A} = x$

$$\delta = 0 = \int_0^{L/2} \frac{R_A x^2}{EI} dx + \int_{L/2}^{L} \frac{R_A x^2 - Px^2 + P\frac{L}{2}x}{EI} dx$$

$$0 = \frac{R_A L^3}{24EI} + \frac{1}{EI}\left[\frac{R_A x^3}{3} - \frac{Px^3}{3} + \frac{PLx^2}{4}\right]_{L/2}^{L} = \frac{R_A L^3}{24EI} + \frac{1}{EI}\left\{\left[\frac{R_A L^3}{3} - \frac{PL^3}{3} + \frac{PL^3}{4}\right] - \left[\frac{R_A L^3}{24} - \frac{PL^3}{24} + \frac{PL^3}{16}\right]\right\}$$

$$\frac{R_A}{3} = \frac{5P}{48}$$

$$R_A = \frac{5P}{16}$$

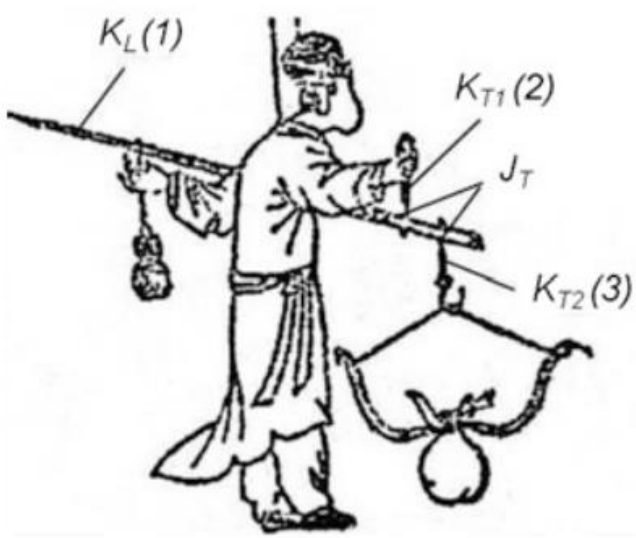
$$R_{By} = P - R_A = \frac{11P}{16}$$

$$M_B = P\frac{L}{2} - R_A L = \frac{3PL}{16}$$

# Ancient Chinese mechanisms

How would you analyse the deflections in these devices?

Quan Heng (權衡)




A water lifting device He Yin (鶴飲)