MEMS1028
Mechanical Design 1

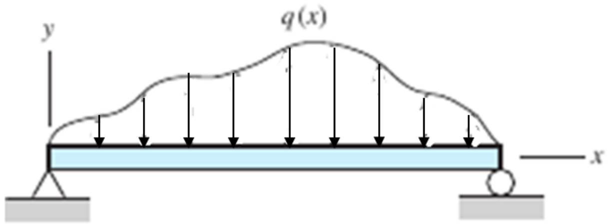
Lecture 5

Advanced deformation analysis (deflection)

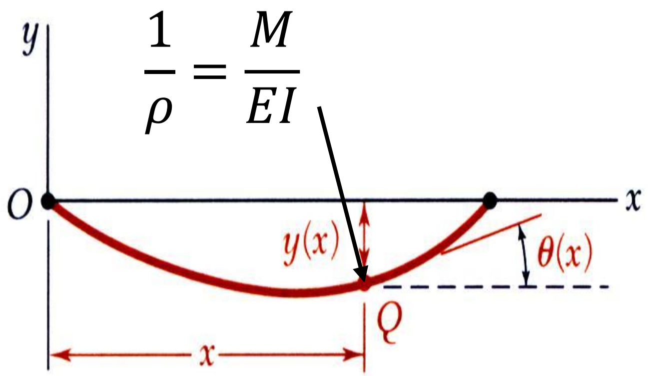
Objectives

- Describe the relationships between shear force and bending moments
- Analyze beam deflections using singularity functions and superposition
- Determine spring rates, spring energy and combining spring rates to determine the characteristics of flexural elements in engineering designs

Shear force & bending moment

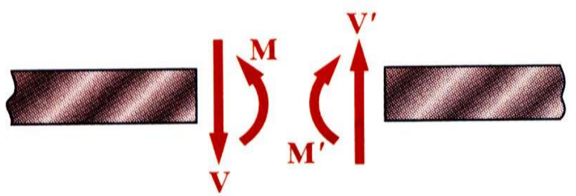


For small θ: $$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \approx \frac{d^2y}{dx^2}$$



$$EI\frac{d^2y}{dx^2} = M$$

$$\frac{dM}{dx} = V \quad \Rightarrow \quad EI\frac{d^3y}{dx^3} = V$$



(a) Internal forces
(positive shear and positive bending moment)

$$\frac{dV}{dx} = -q \quad \Rightarrow \quad EI\frac{d^4y}{dx^4} = -q$$

Deflection due to bending

$$\frac{d^4y}{dx^4} = -\frac{q}{EI}$$

$$\frac{d^3y}{dx^3} = \frac{V}{EI}$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$\frac{dy}{dx} = \theta$$

$$y = f(x)$$

- Deflection = y
- Slope = dy/dx = θ
- Distributed load = q (↓)
- Shear force = V
- Bending moment = M
- All parameters can be functions of x

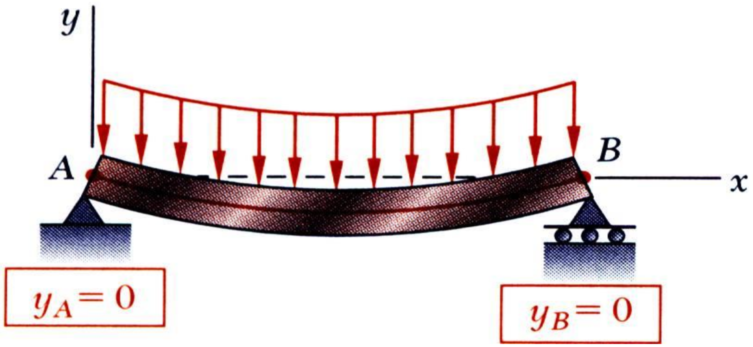
I. The square of the slope of the beam is negligible compared to unity

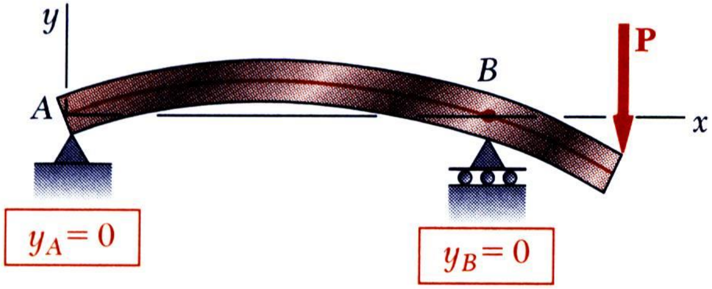
II. The beam deflection due to shearing stresses is negligible (a plane section is assumed to remain plane)

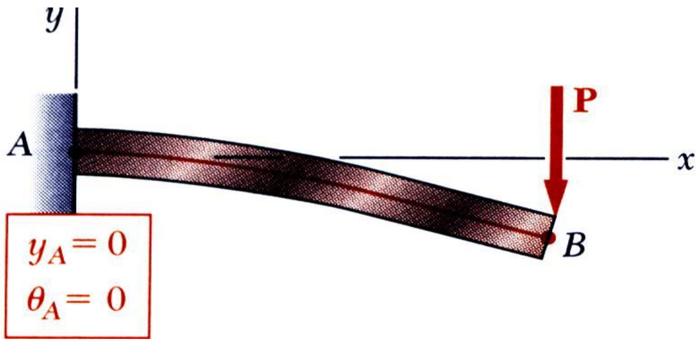
III. The values of E and I remain constant for any interval along the beam

Deflection due to bending

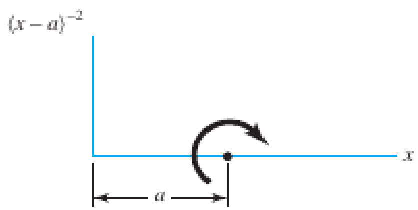
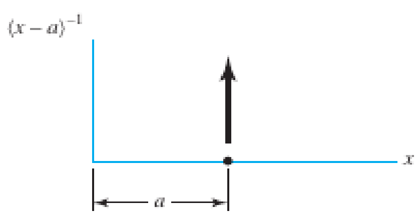
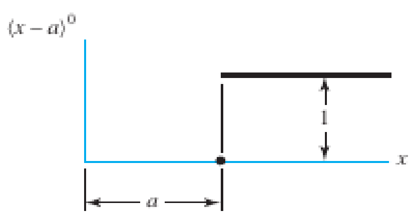
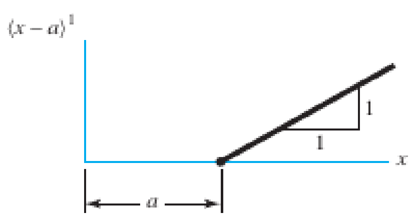
- Integrating constants are determined from boundary conditions; e.g.





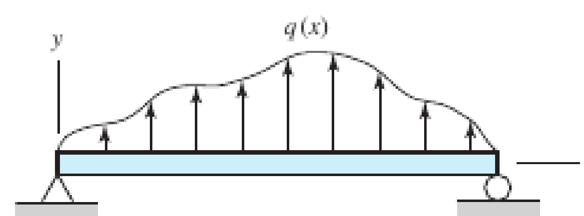


Singularity functions

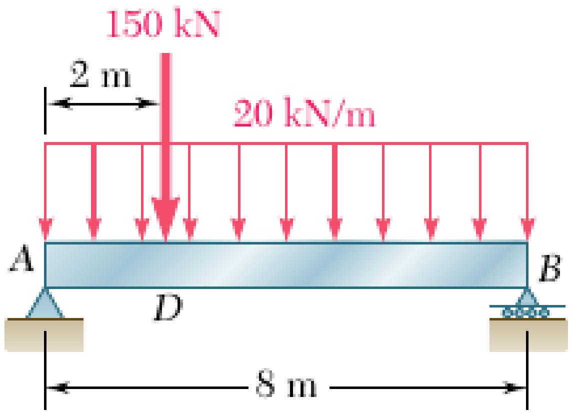
Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x-a\rangle^{-2}=0 \quad x\neq a$ $\langle x-a\rangle^{-2}=\pm\infty \quad x=a$ $\int \langle x-a\rangle^{-2}\,dx=\langle x-a\rangle^{-1}$
Concentrated force (unit impulse)		$\langle x-a\rangle^{-1}=0 \quad x\neq a$ $\langle x-a\rangle^{-1}=+\infty \quad x=a$ $\int \langle x-a\rangle^{-1}\,dx=\langle x-a\rangle^{0}$
Unit step		$\langle x-a\rangle^{0}=\begin{cases}0 & x<a\\ 1 & x\geq a\end{cases}$ $\int \langle x-a\rangle^{0}\,dx=\langle x-a\rangle^{1}$
Ramp		$\langle x-a\rangle^{1}=\begin{cases}0 & x<a\\ x-a & x\geq a\end{cases}$ $\int \langle x-a\rangle^{1}\,dx=\frac{\langle x-a\rangle^{2}}{2}$

†W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics*, vol. 48, pp. 129–130, 1919.

A singularity function is expressed as $\langle x-a\rangle^n$ where n is any integer (positive or negative) including zero, and a is a constant equal to the value of x at the initial boundary of a specific interval along the beam (note: the singularity functions are for loading $q\uparrow$)



Example 1



Determine the slope and displacement at point *D* using singularity functions given $EI = 100\text{MNm}^2$

Draw the FBD and find the reactions at the supports:

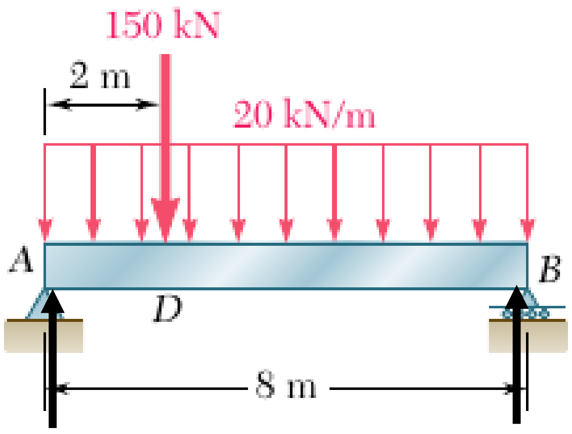
Sum moments about *B*: $\sum M_B = 0$

$$8R_A = 150(6) + 20(8)(4)$$
$$R_A = 192.5\text{kN}\uparrow$$

Sum of vertical forces: $\sum F_y = 0$

$$192.5 + R_B - 150 - 20(8)(4) = 0$$
$$R_B = 117.5\text{kN}\uparrow$$

Example 1



Use tables to express general distributed loads as singularity functions of x for $0 \le x \le L$:

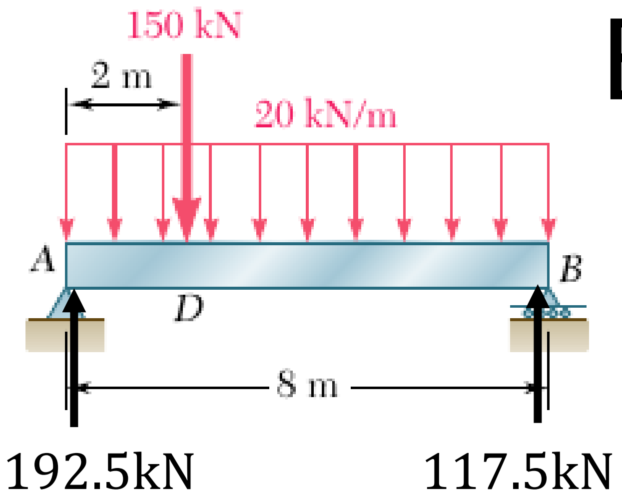
$$q = 192.5(10^3)\langle x\rangle^{-1} - 20(10^3)\langle x\rangle^{0} - 150(10^3)\langle x-2\rangle^{-1} + 117.5(10^3)\langle x-8\rangle^{-1}$$

$$V = 192.5(10^3)\langle x\rangle^{0} - 20(10^3)\langle x\rangle^{1} - 150(10^3)\langle x-2\rangle^{0} + 117.5(10^3)\langle x-8\rangle^{0}$$

$$M = \frac{EI}{10^3}\frac{d^2y}{dx^2} = 192.5\langle x\rangle^{1} - 20\frac{1}{2}\langle x\rangle^{2} - 150\langle x-2\rangle^{1} + 117.5\langle x-8\rangle^{1}$$

$$\frac{EI}{10^3}\frac{dy}{dx} = 192.5\frac{1}{2}\langle x\rangle^{2} - 20\frac{1}{6}\langle x\rangle^{3} - 150\frac{1}{2}\langle x-2\rangle^{2} + 117.5\frac{1}{2}\langle x-8\rangle^{2} + C_1$$

$$\frac{EI}{10^3}y = 192.5\frac{1}{6}\langle x\rangle^{3} - 20\frac{1}{24}\langle x\rangle^{4} - 150\frac{1}{6}\langle x-2\rangle^{3} + 117.5\frac{1}{6}\langle x-8\rangle^{3} + C_1x + C_2$$



Example 1

Evaluate the integration constants using boundary conditions and substitute these into the equations: $0 \le x \le L$

$$\frac{EI}{10^3}y = 192.5\frac{1}{6}\langle x\rangle^3 - 20\frac{1}{24}\langle x\rangle^4 - 150\frac{1}{6}\langle x-2\rangle^3 + 117.5\frac{1}{6}\langle x-8\rangle^3 + C_1 x + C_2$$

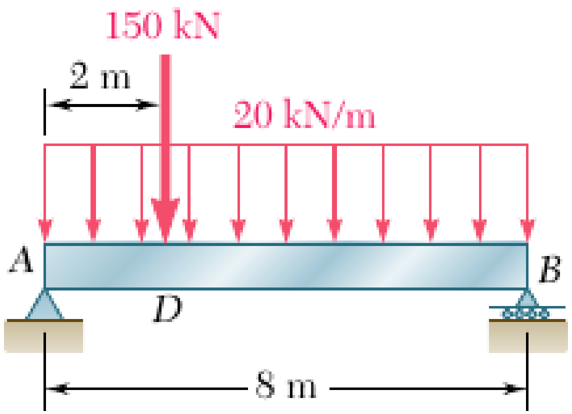
At x=0, y=0: $C_2 = 0$ (Note interpretation of $\langle x\rangle^n$ and $\langle x-2\rangle^n$)

At x=8, y=0: $0 = 192.5\frac{8^3}{6} - 20\frac{8^4}{24} - 150\frac{\langle 8-2\rangle^3}{6} + 117.5\frac{1}{6}\langle 8-8\rangle^3 + 8C_1$

$$C_1 = -951$$

$$\frac{EI}{10^3}y = 192.5\frac{1}{6}\langle x\rangle^3 - 20\frac{1}{24}\langle x\rangle^4 - 150\frac{1}{6}\langle x-2\rangle^3 + 117.5\frac{1}{6}\langle x-8\rangle^3 - 951x$$

Example 1



The slope and displacement given $EI = 100\text{MNm}^2$
At point D (where $x = 2$):

$$\frac{EI}{10^3}\frac{dy}{dx} = 192.5\frac{1}{2}\langle x\rangle^2 - 20\frac{1}{6}\langle x\rangle^3 - 150\frac{1}{2}\langle x-2\rangle^2 + 117.5\frac{1}{2}\langle x-8\rangle^2 - 951$$

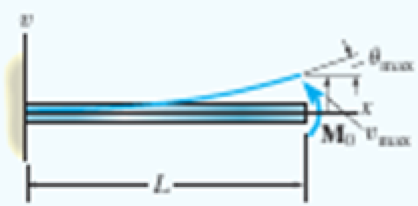
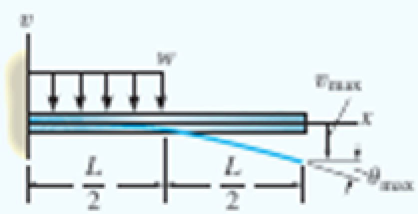
$$\frac{dy}{dx} = \left(\frac{10^3}{100\times10^6}\right)\left(192.5\frac{2^2}{2} - 20\frac{2^3}{6} - 951\right) = -5.93\times10^{-3}\text{rad}$$

$$\frac{EI}{10^3}y = 192.5\frac{1}{6}\langle x\rangle^3 - 20\frac{1}{24}\langle x\rangle^4 - 150\frac{1}{6}\langle x-2\rangle^3 + 117.5\frac{1}{6}\langle x-8\rangle^3 - 951x$$

$$y = \left(\frac{10^3}{100\times10^6}\right)\left(192.5\frac{2^3}{6} - 20\frac{2^4}{24} - 951\times2\right) = -16.6\text{mm}$$

Superposition

Cantilevered Beam Slopes and Deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$v = \frac{-wx^2}{24EI}\left(x^2 - 2Lx + \frac{3}{2}L^2\right)$ $0 \le x \le L/2$ $v = \frac{-wL^3}{384EI}(8x - L)$ $L/2 \le x \le L$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{120EIL}(10L^3 - 10L^2 x + 5Lx^2 - x^3)$

Superposition

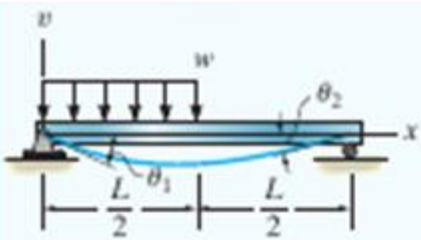
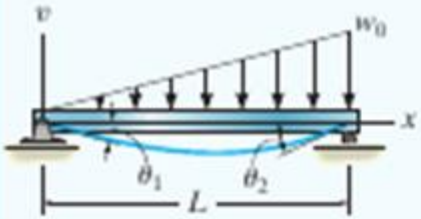
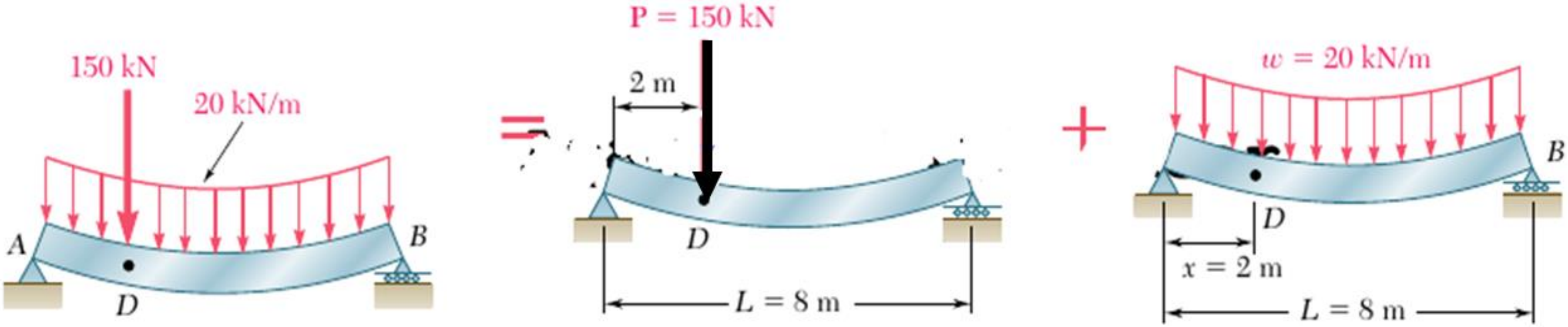
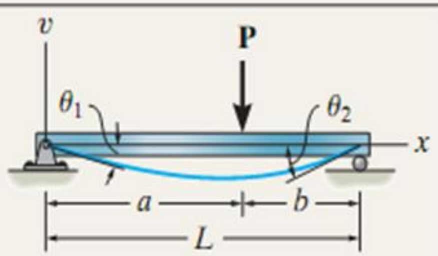
Simply Supported Beam Slopes and Deflections			
Beam	Slope	Deflection	Elastic Curve
	$\theta_1 = \frac{-3wL^3}{128EI}$ $\theta_2 = \frac{7wL^3}{384EI}$	$v\Big\vert_{x=L/2} = \frac{-5wL^4}{768EI}$ $v_{max} = -0.006563\frac{wL^4}{EI}$ at $x = 0.4598L$	$v = \frac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = \frac{-wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x < L$
	$\theta_1 = \frac{-7w_0L^3}{360EI}$ $\theta_2 = \frac{w_0L^3}{45EI}$	$v_{max} = -0.00652\frac{w_0L^4}{EI}$ at $x = 0.5193L$	$v = \frac{-w_0x}{360EIL}(3x^4 - 10L^2x^2 + 7L^4)$

Table A-9 in text has more tables

Example 1

Using superposition:

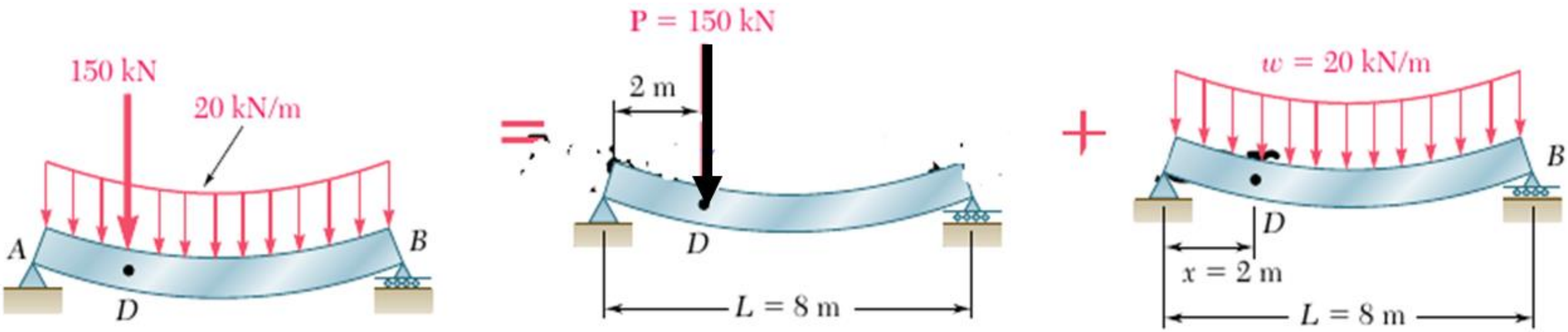


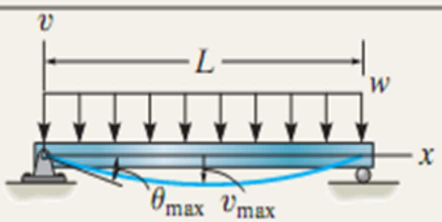
Beam	Slope	Deflection	Elastic Curve
	$\theta_1 = \frac{-Pab(L+b)}{6EIL}$ $\theta_2 = \frac{Pab(L+a)}{6EIL}$	$v\Big\rvert_{x=a} = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$	$v = \frac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$ $0 \leq x \leq a$

$$y_1 = \frac{-150(10^3)(6)(2)}{6(100)(10^6)8}(8^2 - 6^2 - 2^2) = -9\text{mm}$$

Example 1

Using superposition:

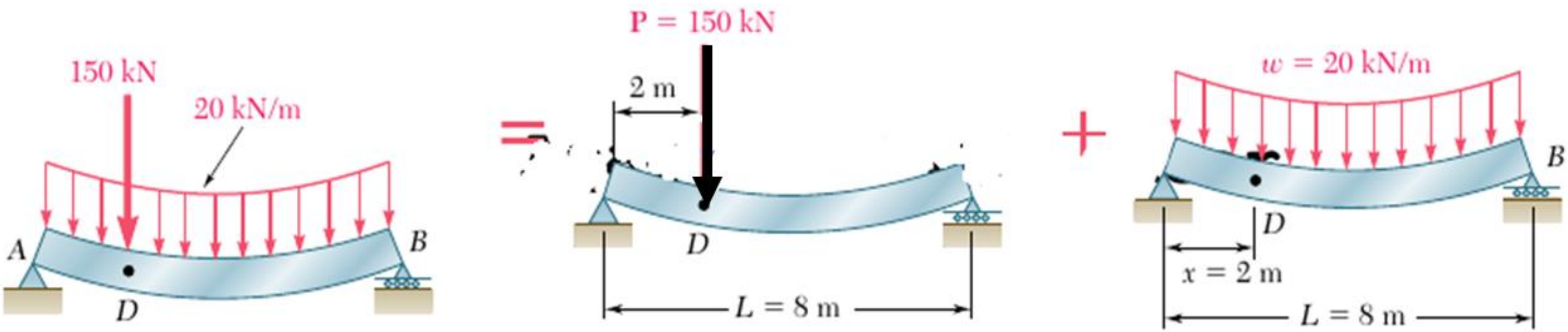


Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{384EI}$	$v = \dfrac{-wx}{24EI}(x^3 - 2Lx^2 + L^3)$

$$y_2 = \frac{-20(10^3)(2)}{24(100)(10^6)}(2^3 - 2(8)2^2 + 8^3) = -7.6\text{mm}$$

Example 1

Using superposition:



$$y_1 = \frac{-Pb}{6EIL}(L^2 x - b^2 x - x^3) \qquad y_2 = \frac{-w}{24EI}(x^4 - 2Lx^3 + L^3 x)$$

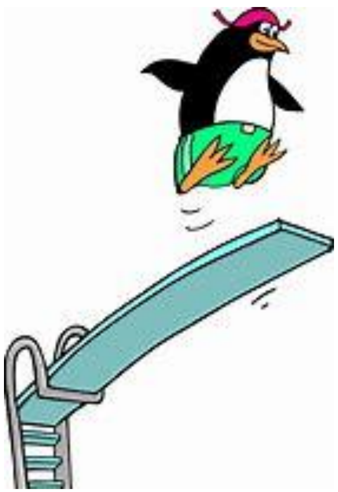
At = 2: $y = y_1 + y_2 = -9 - 7.6 = -16.6\text{mm}$

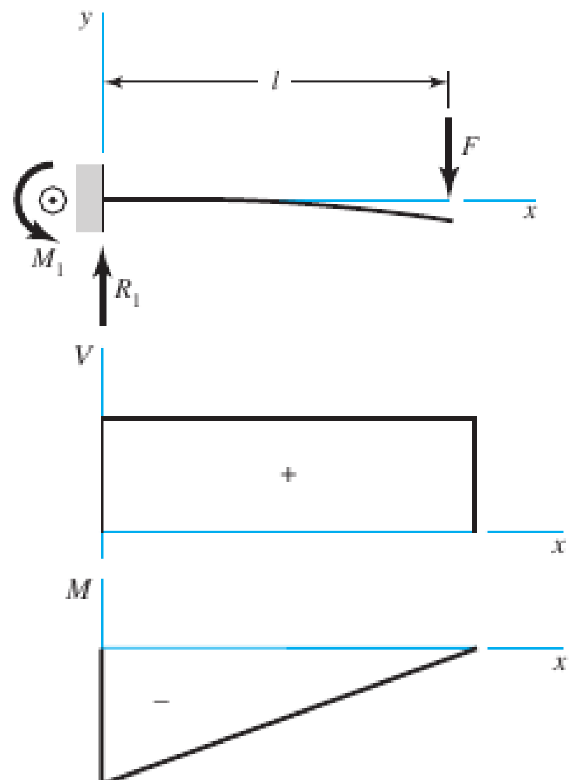
$$\frac{d}{dx}y_1 = \frac{-Pb}{6EIL}(L^2 - b^2 - 3x^2) \qquad \frac{d}{dx}y_2 = \frac{-w}{24EI}(4x^3 - 6Lx^2 + L^3)$$

At = 2: $\theta = \theta_1 + \theta_2 = -0.003 - 0.00293 = -5.93\times10^{-3}\text{rad}$

Spring rates

Beam deflections can be used as spring: e.g. diving board





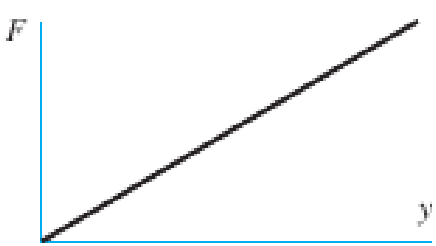
$$R_1 = V = F \qquad M_1 = Fl$$

$$M = F(x - l)$$

$$y = \frac{Fx^2}{6EI}(x - 3l)$$

$$y_{max} = -\frac{Fl^3}{3EI}$$

Assume *F*-*y* is linear at *L*



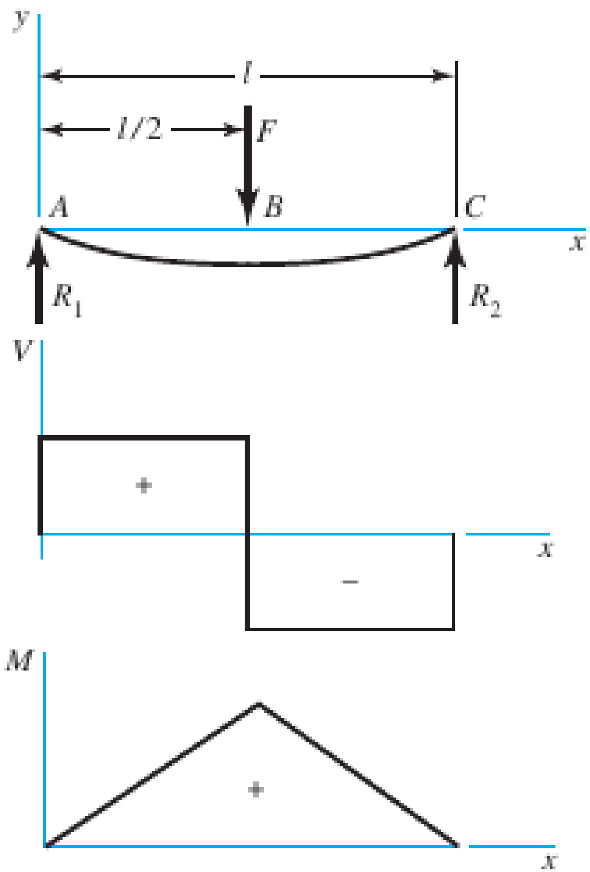
See Appendix A-9

At *L*: spring rate is

$$k = \frac{F}{y} = \frac{3EI}{L^3}$$

Spring rates

Determine the spring rate for the following beam at point *B*:



$$R_1 = R_2 = \frac{F}{2}$$

$$V_{AB} = R_1 \qquad V_{BC} = -R_2$$

$$M_{AB} = \frac{Fx}{2} \qquad M_{BC} = \frac{F}{2}(l - x)$$

$$y_{AB} = \frac{Fx}{48EI}(4x^2 - 3l^2)$$

$$y_{max} = -\frac{Fl^3}{48EI}$$

At *B*: spring rate is

$$k = \frac{F}{y} = \frac{48EI}{L^3}$$

In general: $k = \frac{F}{y} = \frac{\text{force}}{\text{deformation}}$

Spring rates

The total extension or contraction of a uniform bar in pure tension or compression, respectively, is given by

$$\delta = \frac{FL}{AE}$$

What is the equivalent spring rate for an axially loaded bar?

Axially load bar in tension/compression, spring rate is given by

$$k = \frac{F}{\delta} = \frac{AE}{L}$$

Spring rates

The angular deflection of a hollow or solid circular bar under torsion is given by

$$\theta = \frac{TL}{GJ}$$

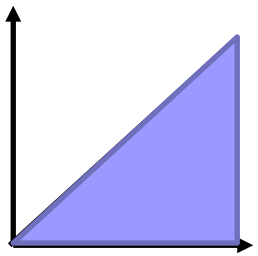
What is the equivalent torsional spring rate for the circular bar?

Circular bar under torsion, spring rate is given by

$$k = \frac{T}{\theta} = \frac{GJ}{L}$$

Spring potential energy

- The work done in deforming the spring is stored as potential energy
- The energy is recovered when the load is removed
- The energy is the area under the force-displacement curve
- For a linear force-displacement, the potential energy is given by



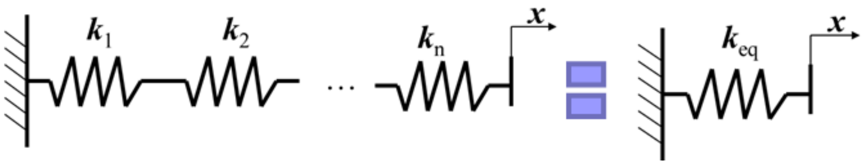
$$U = \frac{1}{2}Fy = \frac{1}{2}ky^2 = \frac{1}{2}F\frac{F}{k} = \frac{F^2}{2k}$$

- This potential energy causes the deformation or strain and is also know as the strain energy

Springs combination

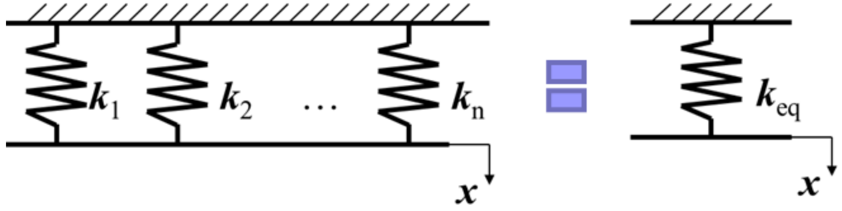
- The spring rate changes when springs are connected in series and parallel
- For "*n*" springs with rates k_1, k_2, … k_n connected in series, the equivalent spring rate k_{eq} is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$



- For "*n*" springs with rates k_1, k_2, … k_n connected in parallel, the equivalent spring rate k_{eq} is given by

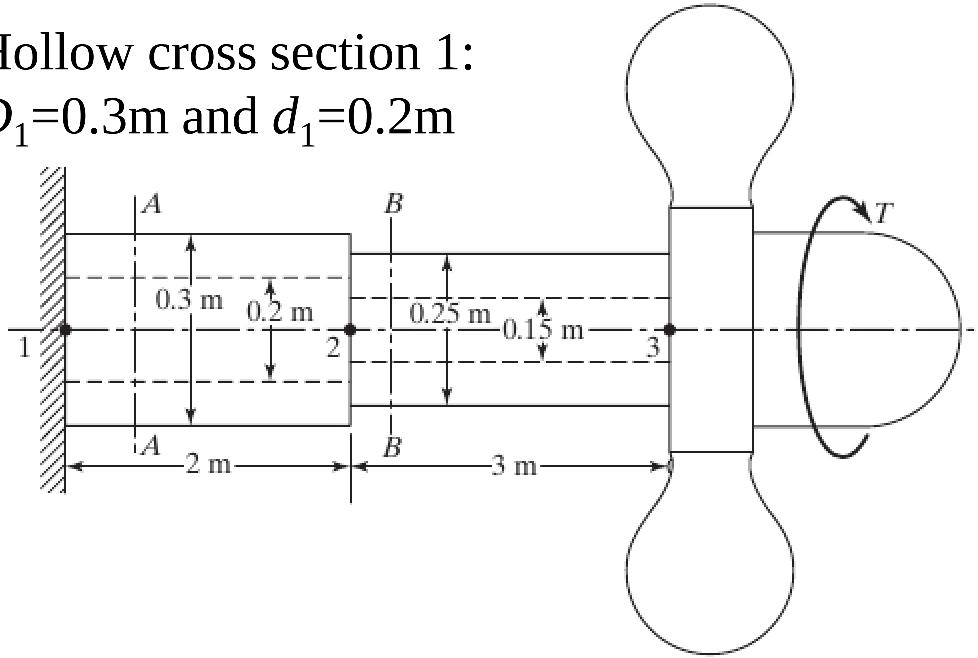
$$k_{eq} = k_1 + k_2 + \cdots + k_n$$



Example 2

Determine the torsional spring constant of the steel propeller shaft shown

Hollow cross section 1:
D_1=0.3m and d_1=0.2m



Hollow cross section 2:
D_2=0.25m and d_2=0.15m

Circular bar under torsion,

$$k = \frac{T}{\theta} = \frac{GJ}{L} = \frac{G\pi(D^4 - d^4)}{32L}$$

$$k_1 = \frac{80 \times 10^9 \pi(0.3^4 - 0.2^4)}{32(2)}$$

$$k_1 = 25.5255 \text{MNm/rad}$$

$$k_2 = \frac{80 \times 10^9 \pi(0.25^4 - 0.15^4)}{32(3)}$$

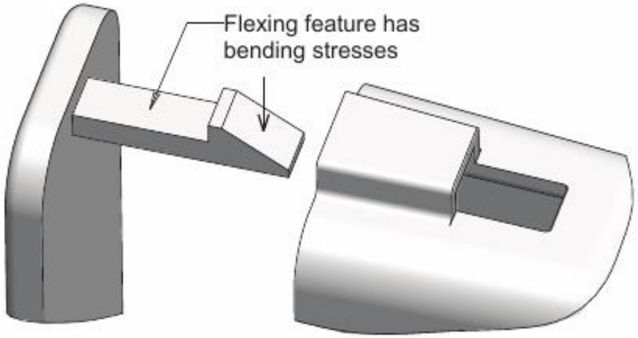
$$k_2 = 8.9012 \text{MNm/rad}$$

Spring connected in series

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$$

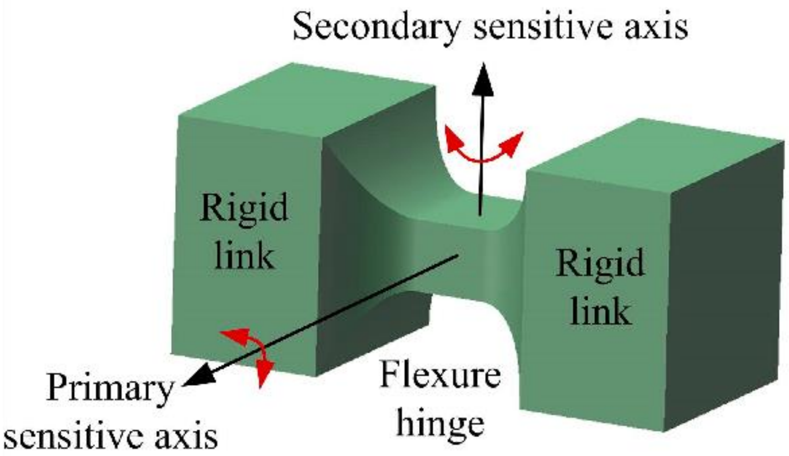
$$k_{eq} = 6.6 \text{MNm/rad}$$

Applications of flexural elements



Buckles

Leaf spring in trailer



Design analysis

How would you analyse the spring rates for the following chair design?