MEMS1028
Mechanical Design 1

Lecture 3 part 2

Load & stress analysis (Design of power transmitting shafts)

Objectives

- Discuss the design of power transmitting shafts with combined loadings
- Introduce and compare shaft design standards

Introduction

- Shafts are used for transmitting power P (in Watts) or torque T (in Nm) and are commonly found in rotating machines
- Shaft power transmission: $P = T\omega$
- Angular frequency: $\omega = 2\pi f$ (rad/s) and frequency f in Hz (rev/s)

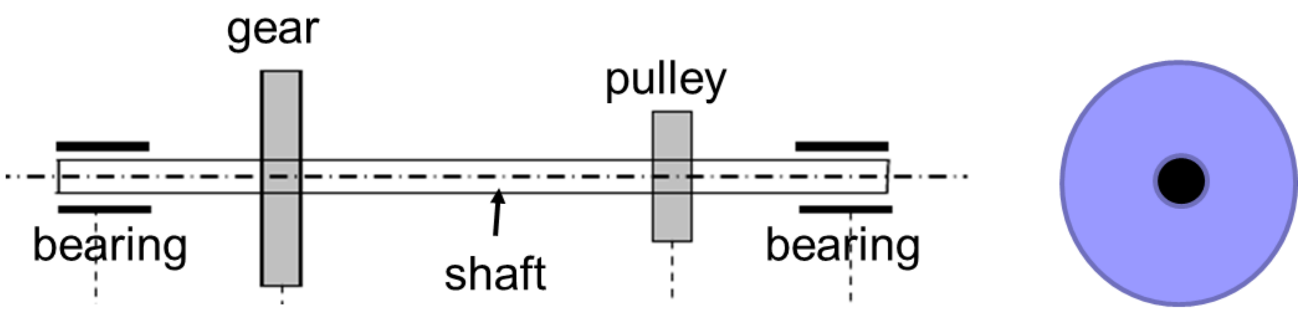
Note: $1\,\frac{\text{rev}}{\text{min}} = \frac{2\pi}{60}\frac{\text{rad}}{\text{s}}$

For a shaft transmitting power P_o (in kW) at rotational speed η (in rpm):

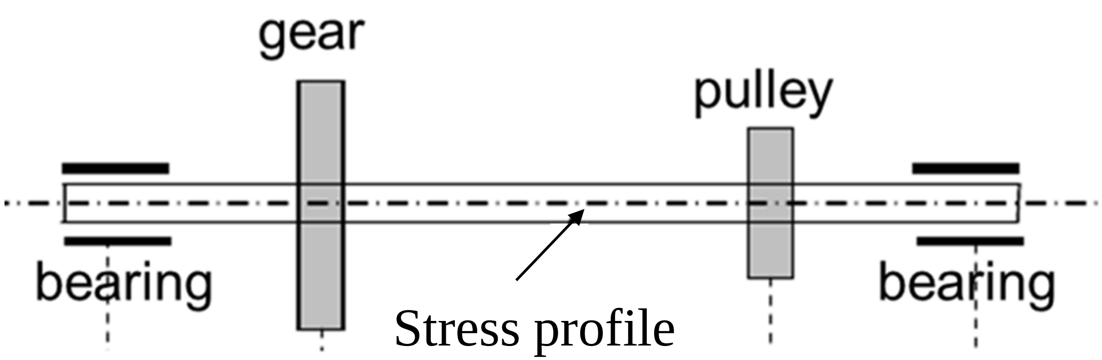
$$T\ (\text{Nm}) = 9550\,\frac{P_o\ (\text{kW})}{\eta\ (\text{rpm})}$$

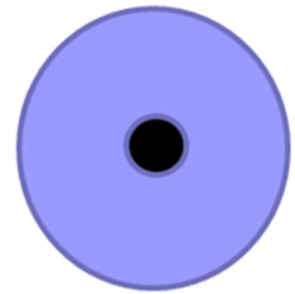
Introduction

- The design of these shafts involved finding the shaft dimensions based on strength considerations
- These shafts are usually subjected to static and fluctuating loads (dynamic loadings will be covered later in the course)
- The loadings are mainly torsional and bending (sometimes there can be axial loading)

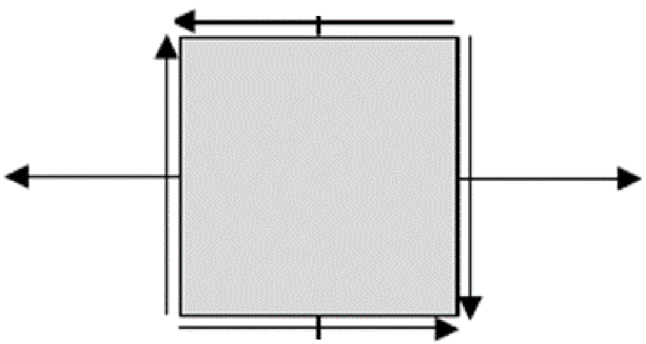


Introduction





Stress profile
at point *B*



Bending stress
$\sigma = -\frac{My}{I}$

Shear stress $\tau = \frac{T\rho}{J}$

❖ The magnitude of the shear stress and bending stress changes with locations

Maximum bending stress

For circular "hollow" shaft with:

- Outside diameter d_o
- Inside diameter d_i
- Moment of inertia is $I = \frac{\pi}{64}\left(d_o^4 - d_i^4\right)$
- At the surface of the shaft $y = d_o/2$, the magnitude of the bending stress is maximum

$$\sigma = \frac{My}{I} = \frac{M\frac{d_o}{2}}{\frac{\pi}{64}\left(d_o^4 - d_i^4\right)} = \frac{32M}{\pi d_o^3(1-\alpha^4)} = \frac{B}{\pi d_o^3}(32M)$$

Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$

- For solid shaft, $B = 1$

Maximum shear stress

For circular “hollow” shaft with:

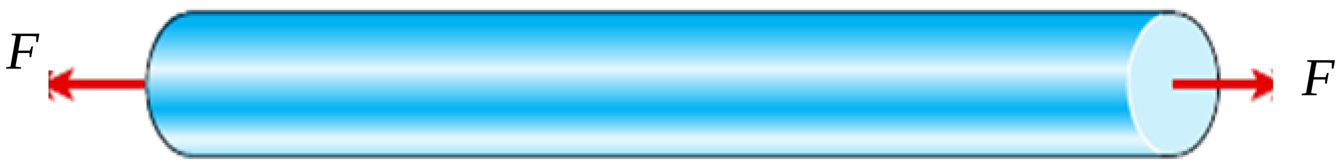
- ❖ Outside diameter d_o
- ❖ Inside diameter d_i
- ❖ Polar moment of inertia is $J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) = 2I$ where I = area moment of inertia
- ❖ At the surface of the shaft $\rho = d_o/2$, the magnitude of the shear stress is maximum along the neutral axis

$$\tau = \frac{T\rho}{J} = \frac{T\frac{d_o}{2}}{\frac{\pi}{32}\left(d_o^4 - d_i^4\right)} = \frac{16T}{\pi d_o^3(1-\alpha^4)} = \frac{B}{\pi d_o^3}(16T)$$

Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$

- ❖ For solid shaft, $B = 1$

Axial loading



For circular “hollow” shaft subjected to an axial force *F* (can be tension or compression) with:

- Outside diameter d_o
- Inside diameter d_i
- The axial stress is

$$\sigma_a = \frac{F}{\frac{\pi}{4}\left(d_o^2 - d_i^2\right)} = \frac{4F}{\pi d_o^2(1-\alpha^2)} = \frac{4F(1+\alpha^2)}{\pi d_o^2(1-\alpha^2)(1+\alpha^2)}$$

$$\sigma_a = \frac{4Fd_o(1+\alpha^2)}{\pi d_o^3(1-\alpha^4)} = \frac{4Fd_o(1+\alpha^2)}{\pi d_o^3(1-\alpha^4)}$$

$$\sigma_a = \frac{4B}{\pi d_o^3}\left(Fd_o(1+\alpha^2)\right)$$

Case 1

Pure torque:

- For a shaft transmitting power P_o (in kW) at rotational speed η (in rpm):

$$T\ (\text{Nm}) = \frac{P_o\ (\text{kW})}{\eta\ (\text{rpm})}$$

- For a solid circular shaft, the nominal stress is

$$\tau = \frac{16T}{\pi d_o^3}$$

- For a hollow circular shaft, the nominal stress is

$$\tau = \frac{16T}{\pi d_o^3} B$$

- Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$

Case 2

Pure bending:

❖ For a solid circular shaft, the nominal stress is

$$\sigma = \frac{32M}{\pi d_o^3}$$

❖ For a hollow circular shaft, the nominal stress is

$$\sigma = \frac{32M}{\pi d_o^3} B$$

❖ Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$

Case 2a

Combined bending with torsion:

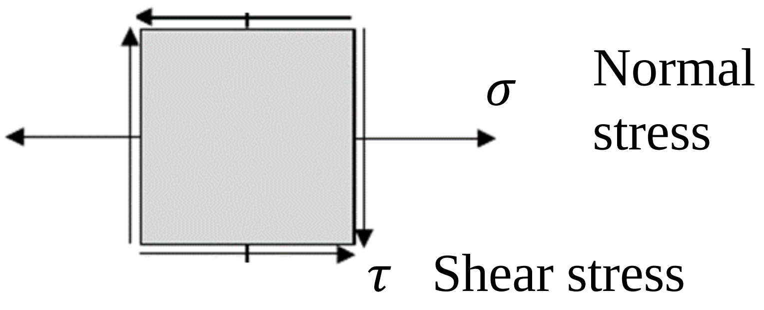
- For bending, the normal stress is

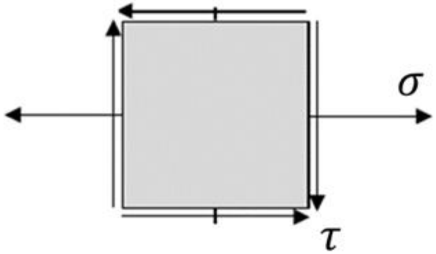
$$\sigma = \frac{32M}{\pi d_o^3} B$$

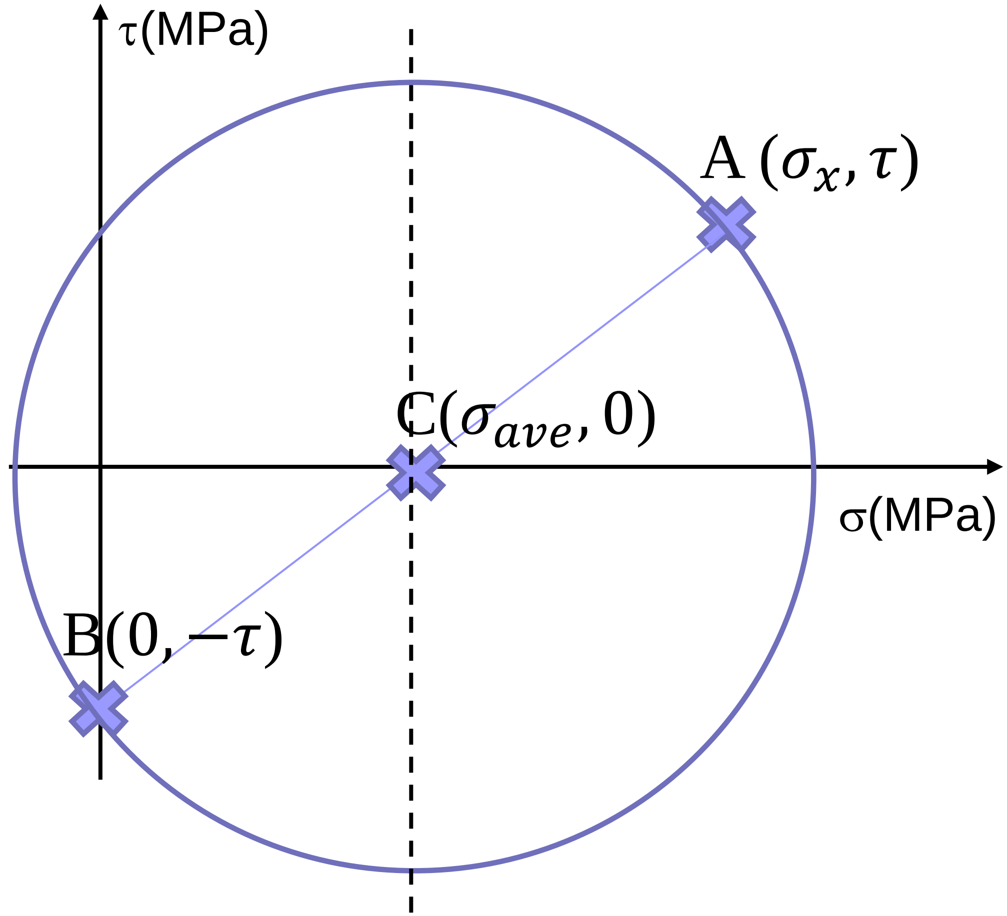
- For torsion, the shear stress is

$$\tau = \frac{16T}{\pi d_o^3} B$$

- Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$







$$\tau = \frac{16T}{\pi d_o^3}B$$

$$\sigma_x = \frac{32M}{\pi d_o^3}B$$

$$\sigma_y = 0$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x}{2}$$

$$\sigma_{ave} = \frac{16M}{\pi d_o^3}B$$

- Circle radius (i.e. the max shear)

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau^2} = \frac{16B}{\pi d_o^3}\{M^2 + T^2\}^{1/2}$$

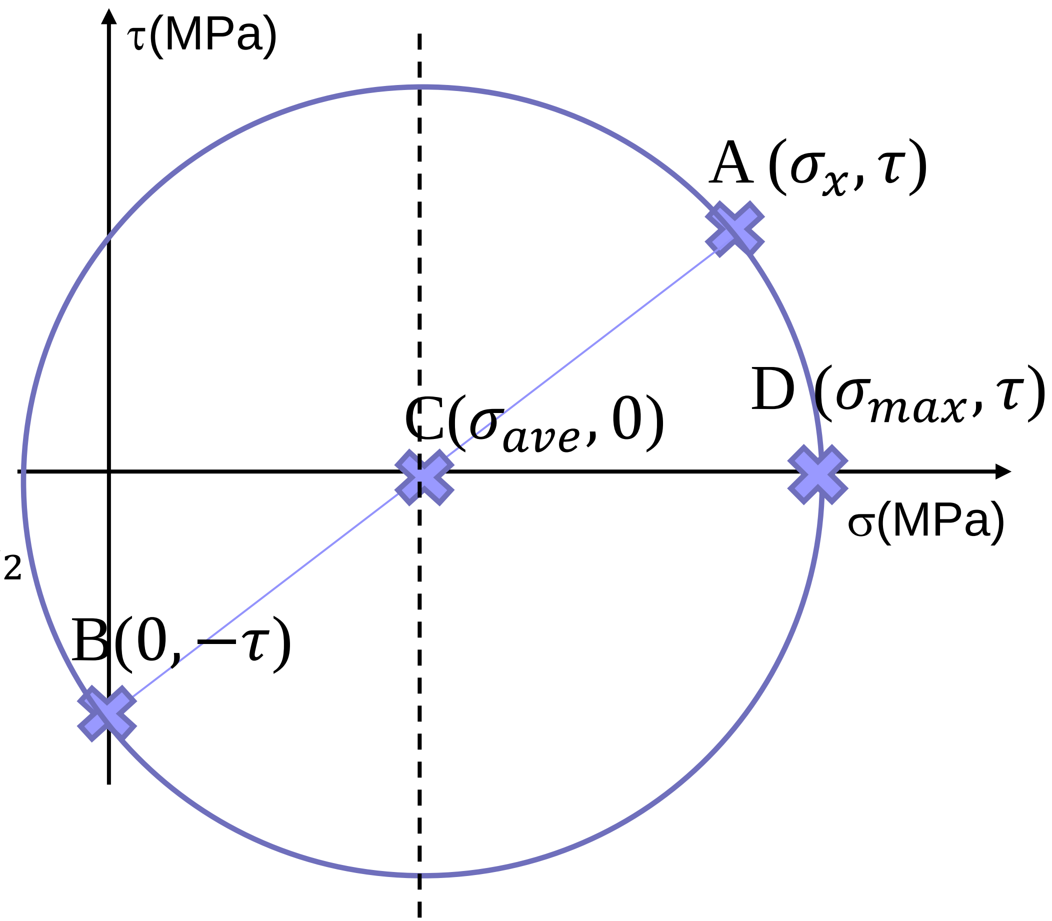
Note that the max normal stress is at point D where

$$\sigma_{max} = \sigma_{ave} + \tau_{max}$$

$$\sigma_{max} = \frac{16M}{\pi d_o^3} B + \frac{16B}{\pi d_o^3} \{M^2 + T^2\}^{1/2}$$

$$\sigma_{max} = \frac{32B}{\pi d_o^3} \left\{ \frac{1}{2} \left(M + \sqrt{M^2 + T^2} \right) \right\}$$

$$\sigma_{max} = \frac{32B}{\pi d_o^3} \{M_e\}$$



Note: $M_e = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right)$ is called the equivalent bending moment

Similarly, $\tau_{max} = \frac{16B}{\pi d_o^3} T_e$ where $T_e = \sqrt{M^2 + T^2}$ is called the equivalent torque

Case 3

Combined torsion, bending and axial loading:

- The bending and axial stresses are normal stresses and can be added. The maximum normal stress is

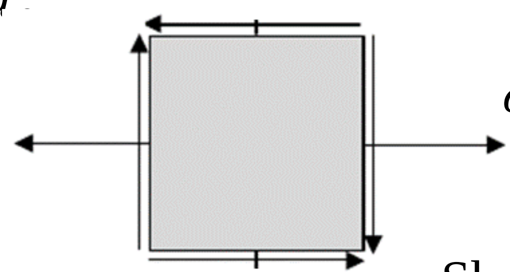
$$\sigma_x = \frac{32B}{\pi d_o^3} M + \frac{4B}{\pi d_o^3}\left(F d_o (1+\alpha^2)\right) = \frac{32B}{\pi d_o^3} M + \frac{32B}{\pi d_o^3}\left(\frac{F d_o (1+\alpha^2)}{8}\right)$$

$$\sigma_x = \frac{32B}{\pi d_o^3}\left(M + \frac{F d_o (1+\alpha^2)}{8}\right)$$

- The shear stress due to torsion is

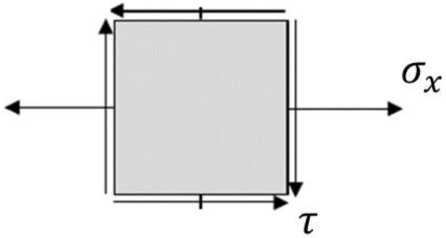
$$\tau = \frac{16T}{\pi d^3} B$$

- Note: $\alpha = \frac{d_i}{d_o}$ and $B = \frac{1}{1-\alpha^4}$



σ_x Normal stress

τ Shear stress



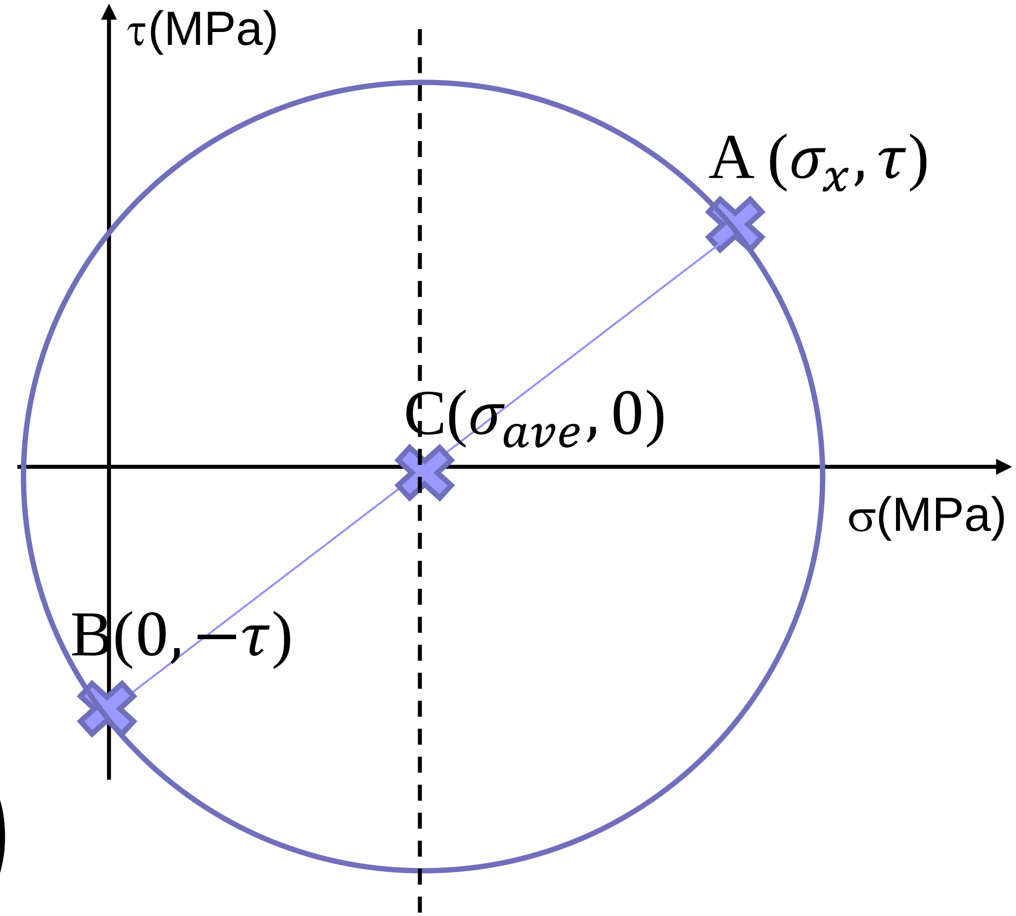
$$\tau = \frac{16T}{\pi d_o^3} B$$

$$\sigma_x = \frac{32}{\pi d_o^3} B \left(M + \frac{F d_o (1 + \alpha^2)}{8} \right)$$

$$\sigma_y = 0$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_{ave} = \frac{16}{\pi d_o^3} B \left(M + \frac{F d_o (1 + \alpha^2)}{8} \right)$$



- Circle radius (i.e. the max shear)

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau^2} = \frac{16B}{\pi d_o^3} \left\{ \left(M + \frac{F d_o (1 + \alpha)}{8} \right)^2 + T^2 \right\}^{1/2}$$

Case 3

$$\tau_{max} = \frac{16B}{\pi d_o^3}\left\{\left(M + \frac{Fd_o(1+\alpha)}{8}\right)^2 + T^2\right\}^{1/2} = \frac{16B}{\pi d_o^3}(T_e)$$

Note: $T_e = \sqrt{\left(M + \frac{Fd_o(1+\alpha)}{8}\right)^2 + T^2}$ is called the equivalent torque

The equivalent bending moment which has the form $M_e = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right)$ is

$$M_e = \frac{1}{2}\left(M + \frac{Fd_o(1+\alpha)}{8} + \sqrt{\left(M + \frac{Fd_o(1+\alpha)}{8}\right)^2 + T^2}\right)$$

Case 3 (ASME)

Old ASME design code: based on max shear stress

$$\tau_{allow} = \frac{16B}{\pi d_o^3}\left\{\left(\mathrm{C_{bm}}M + \frac{\mathrm{C_{ca}}F d_o(1+\alpha)}{8}\right)^2 + \mathrm{C_t}T^2\right\}^{1/2}$$

- $\mathrm{C_{bm}}$ = bending factor; $\mathrm{C_{ca}}$ = column action factor, $\mathrm{C_t}$ = torsion factor

	$\mathbf{C_{bm}}$	$\mathbf{C_t}$
For stationary shaft:		
Load gradually applied	1.0	1.0
Load suddenly applied	1.5 - 2.0	1.5 - 2.0
For rotating shaft:		
Load gradually applied	1.5	1.0
Load suddenly applied (minor shock)	1.5 - 2.0	1.0 - 1.5
Load suddenly applied (heavy shock)	2.0 - 3.0	1.5 - 3.0

Case 3 (ASME)

- Column action factor arises due the phenomenon of buckling of long slender members which are acted upon by axial compressive loads (for tensile load, $C_{ca} = 1$)

$$C_{ca} = \frac{1}{0.0044(L/k)} \text{ for } L/k < 115$$

$$C_{ca} = \frac{\sigma_{yc}}{\pi^2 n E}\left(\frac{L}{k}\right)^2 \text{ for } L/k \geq 115$$

Note: L = shaft length, k = radius of gyration, E = Young's modulus

σ_{yc} = yield stress in compression, and n depends on end constraints

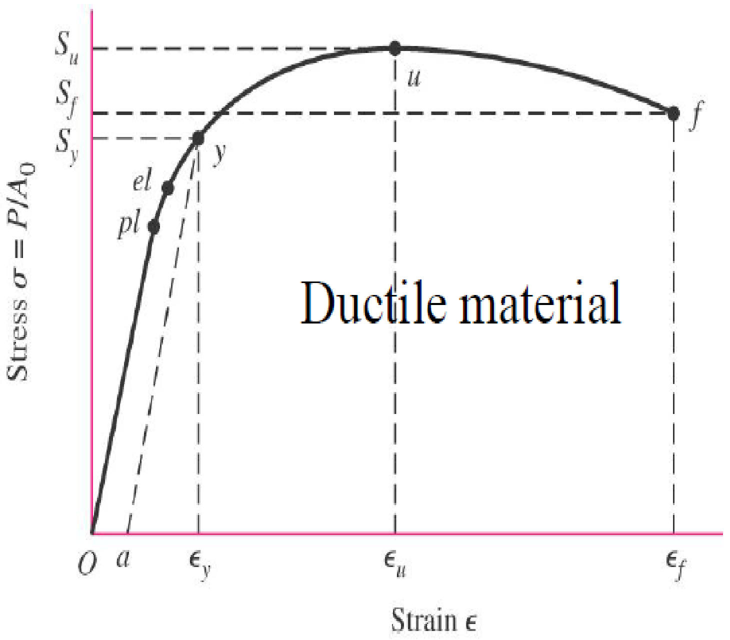
- $n = 1$ for hinged ends
- $n = 2.25$ for fixed ends
- $n = 1.6$ for partly restrained ends as in bearings

Case 3 (ASME)

- ASME code also suggests about the allowable design stress, τ_{allow} to be considered for steel shafting
- ASME Code for commercial steel shafting
- for shaft without keyway $\tau_{allow} = 55$ MPa
- for shaft with keyway $\tau_{allow} = 40$ MPa
- To give you an idea on the factor of safety:
- Typical values of the ultimate strength of low carbon steel ranges from 400 to 600 MPa and the shear strength ranges from 240 to 360 MPa;

Case 3 (ASME)

- ASME Code for steel purchased under definite specifications $\tau_{allow} = 30\%$ of the yield strength but not over 18% of the ultimate strength in tension for shafts without keyways. These values are to be reduced by 25% for the presence of keyways
- To give you an idea on the factor of safety: Typical shear strength is about 60% of ultimate strength



Case 3 (NASA ref)

NASA ref: old ANSI/ASME code: based on max distortion energy theory

$$\sigma_{allow} = \frac{32B}{\pi d_o^3}\left\{\left(M + \frac{F d_o(1+\alpha)}{8}\right)^2 + \frac{3}{4}T^2\right\}^{1/2}$$

- ❖ Consider ductile and brittle materials separately
- ❖ For ductile materials: assume failure occurs elastic failure at yield strength σ_y with negligible effect of stress concentration for static loading so that safe shaft diameter can be found using

$$d_o^3 = \left(\frac{\mathrm{FS}}{\sigma_y}\right)\frac{32B}{\pi}\left\{\left(M + \frac{F d_o(1+\alpha)}{8}\right)^2 + \frac{3}{4}T^2\right\}^{1/2}$$

- ❖ Recommended factor of safety FS > 1.5 to 6

Case 3 (NASA ref)

❖ For brittle materials: assume plastic failure due to cracking at ultimate strength σ_u with application of stress concentration factors for torsion, bending and axial loadings so that safe shaft diameter can be found using

$$d_o^3 = \left(\frac{\text{FS}}{\sigma_u}\right)\frac{32B}{\pi}\left\{\left((K_t)_b M + (K_t)_a \frac{F d_o(1+\alpha)}{8}\right)^2 + \frac{3}{4}\left((K_t)_t T\right)^2\right\}^{1/2}$$

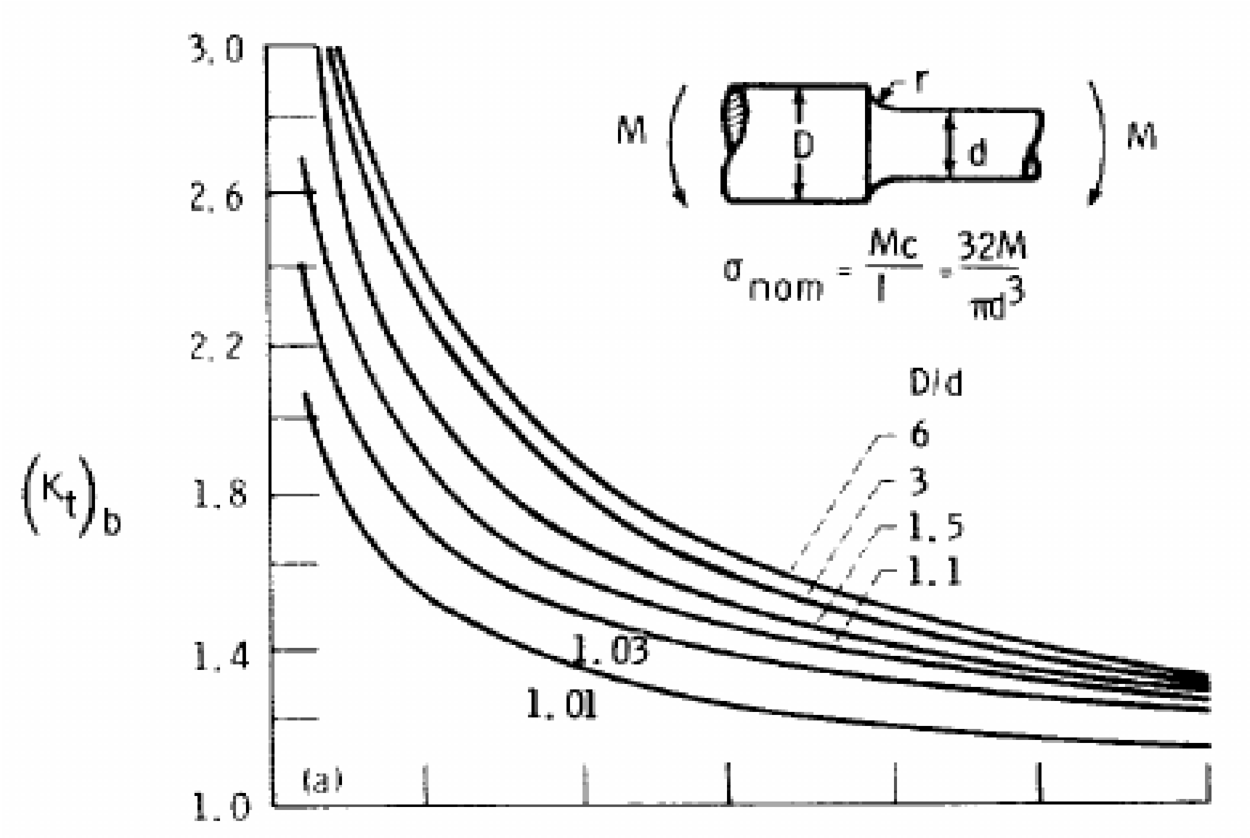
Note:

- $(K_t)_b$ = theoretical stress concentration factor in bending
- $(K_t)_a$ = theoretical stress concentration factor in axial loading
- $(K_t)_t$ = theoretical stress concentration factor in torsion

❖ We will cover these in more details next week

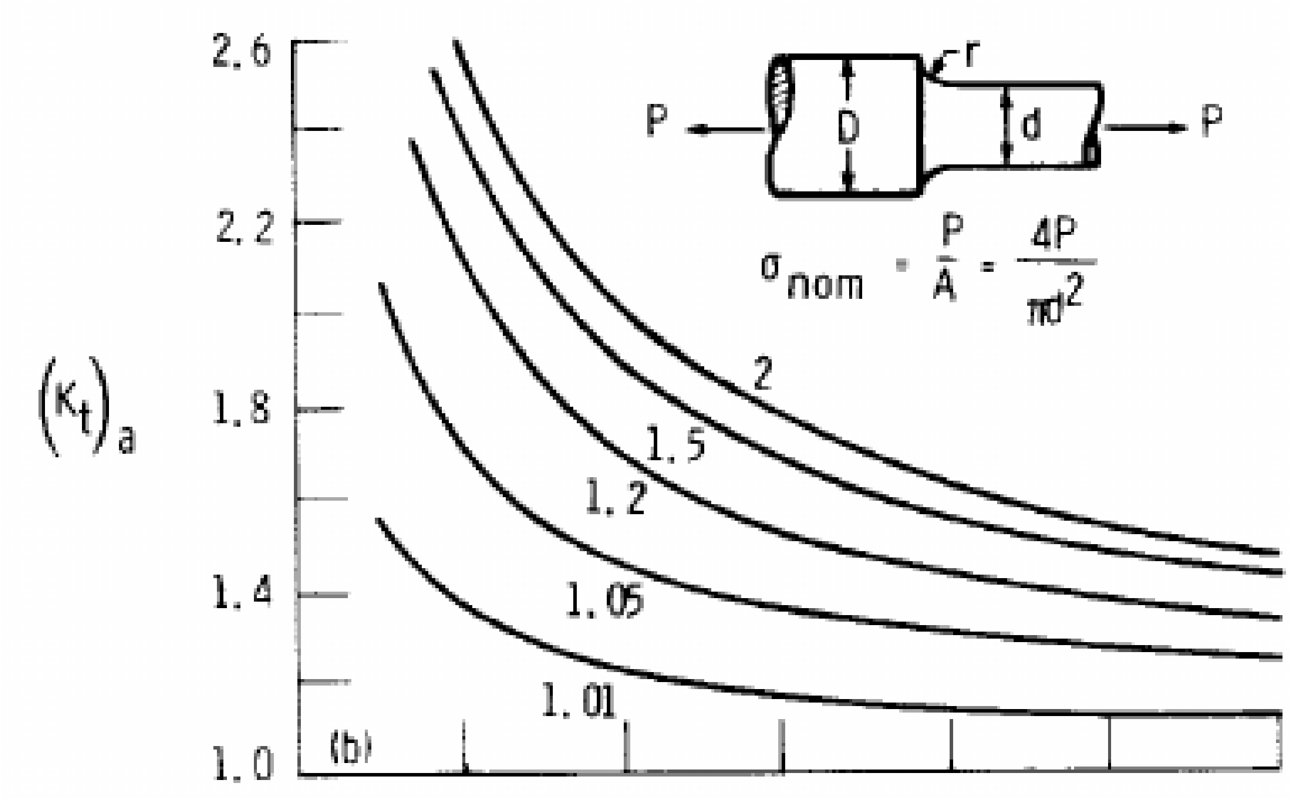
Case 3 (NASA ref)

- $(K_t)_b$ = theoretical stress concentration factor in bending



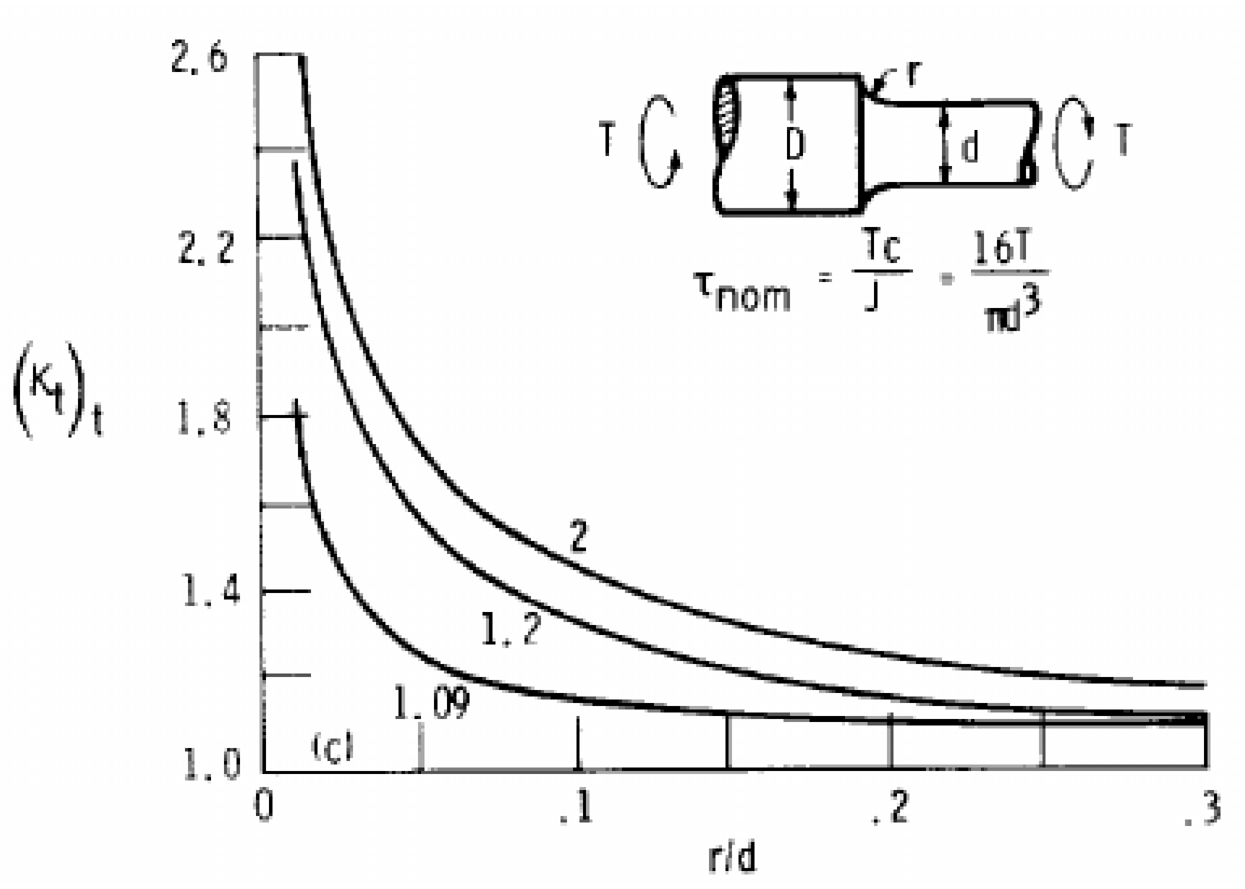
Case 3 (NASA ref)

- $(K_t)_a$ = theoretical stress concentration factor in axial loading



Case 3 (NASA ref)

- $(K_t)_t$ = theoretical stress concentration factor in torsion



Case 4 (NASA ref)

Pure torsion for short shafts

- For short, solid shafts having only transverse shear loading, shaft diameter is given by

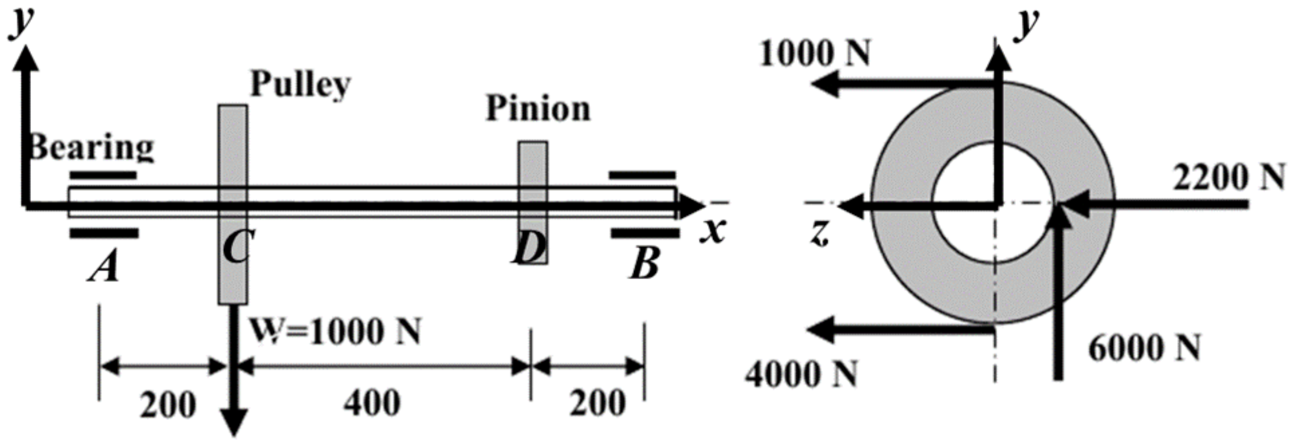
$$d_o = 1.7V\left(\frac{\mathrm{FS}}{\tau_y}\right)$$

Note:

- V = maximum transverse shear load
- τ_y = shear yield strength $= 0.577\sigma_y$ for most steels

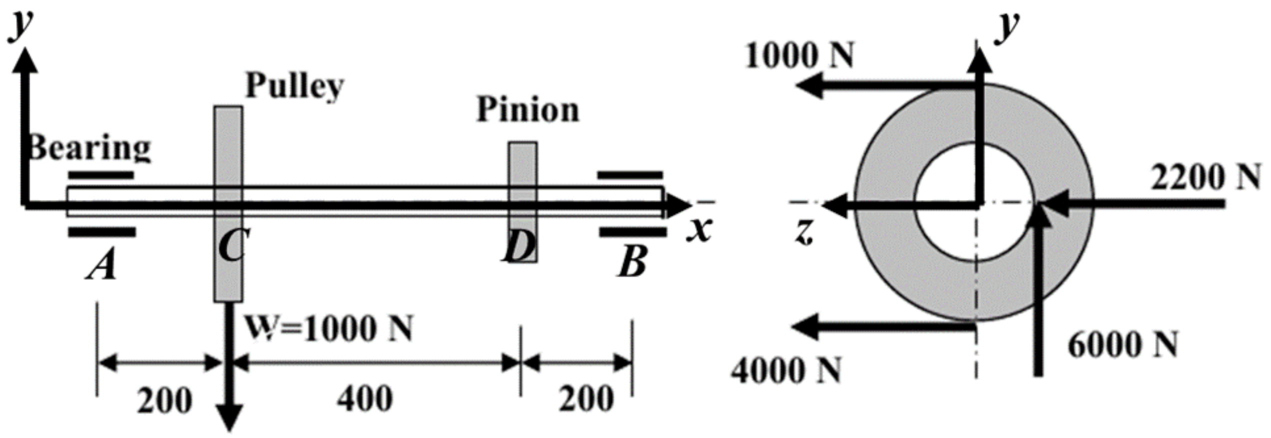
Example 1

A pulley drive is transmitting power to a pinion, which in turn is transmitting power to some other machine element. Pulley and pinion diameters are 400mm and 200mm respectively. Design the shaft for minor to heavy shock using the old ASME code with $C_{bm} = 2$ and $C_t = 1.5$ (all dimensions in mm)

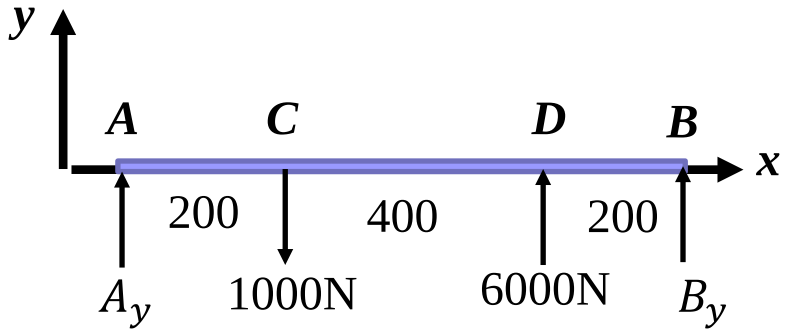


- Note: Torque is $T = (4000 - 1000)0.2 = 600$ Nm
- Loads acting on the shaft are in both horizontal and vertical planes

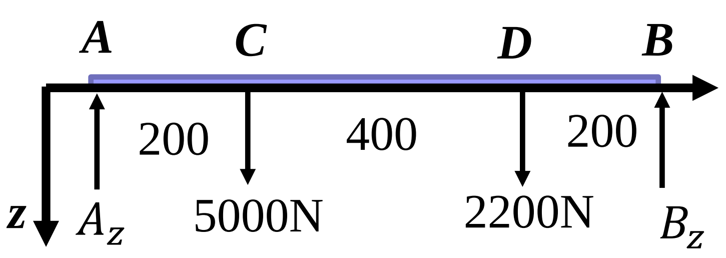
Example 1



Consider the forces in the vertical and horizontal planes:

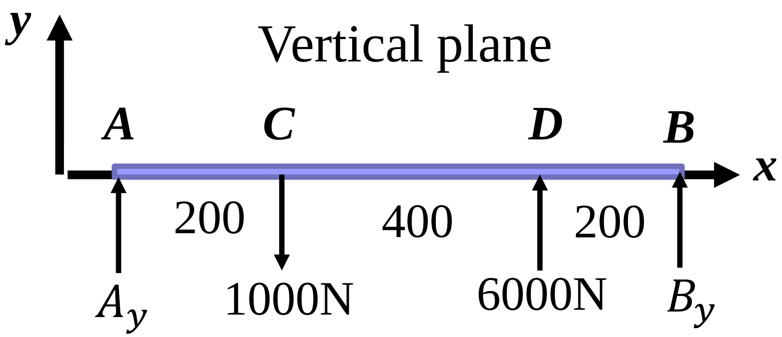


Vertical plane



Horizontal plane

Example 1



Reactions:

$A_y = -750$ N; $B_y = -4250$ N;

❖ Using singularity functions (refer to Table 3.1)

$$q(x) = -750\langle x\rangle^{-1} - 1000\langle x-0.2\rangle^{-1} + 6000\langle x-0.6\rangle^{-1} - 4250\langle x-0.8\rangle^{-1}$$

❖ Integrate to get shear force

$$V(x) = -750\langle x\rangle^{0} - 1000\langle x-0.2\rangle^{0} + 6000\langle x-0.6\rangle^{0} - 4250\langle x-0.8\rangle^{0}$$

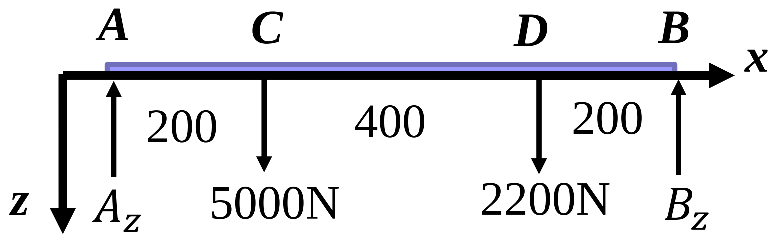
❖ Integrate to get bending moment

$$M(x) = -750\langle x\rangle^{1} - 1000\langle x-0.2\rangle^{1} + 6000\langle x-0.6\rangle^{1} - 4250\langle x-0.8\rangle^{1}$$

The shear force and bending moment diagrams can be determined

Example 1

Horizontal plane



Reactions:

$A_z = 4300$ N; $B_y = 2900$ N;

❖ Using singularity functions (refer to Table 3.1)

$q(x) = 4300\langle x\rangle^{-1} - 5000\langle x-0.2\rangle^{-1} - 2200\langle x-0.6\rangle^{-1} + 2900\langle x-0.8\rangle^{-1}$

❖ Integrate to get shear force

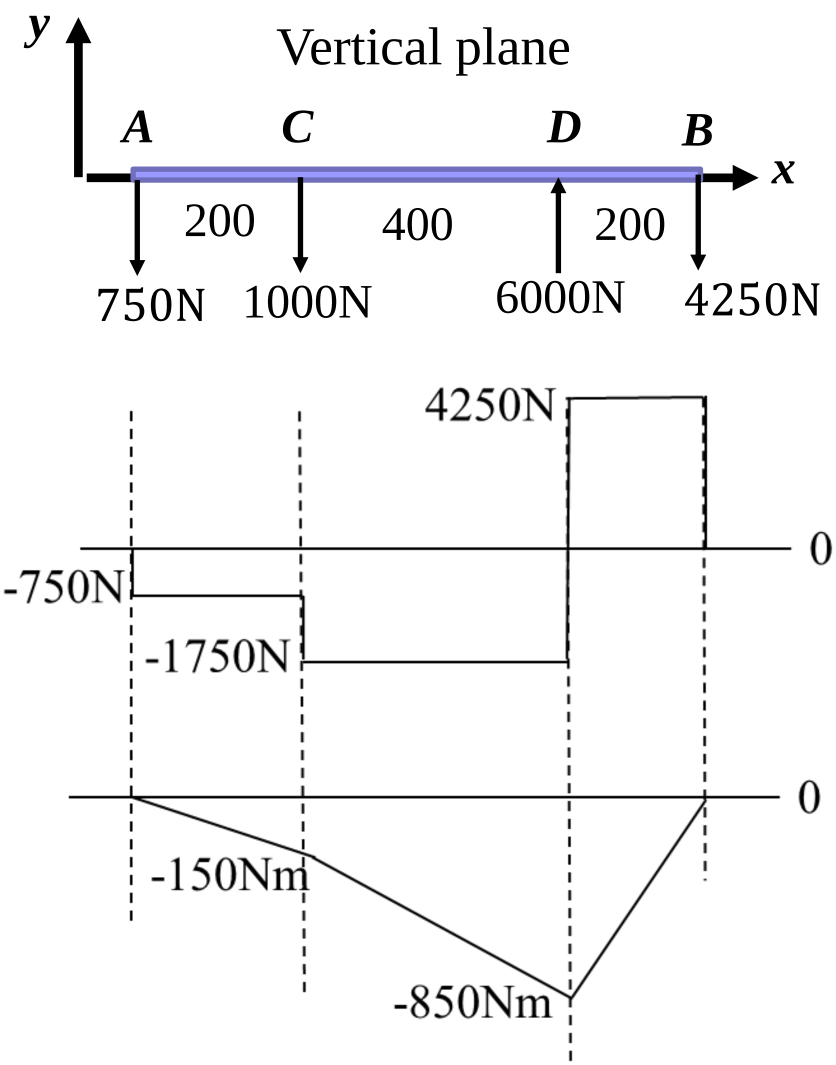
$V(x) = 4300\langle x\rangle^{0} - 5000\langle x-0.2\rangle^{0} - 2200\langle x-0.6\rangle^{0} + 2900\langle x-0.8\rangle^{0}$

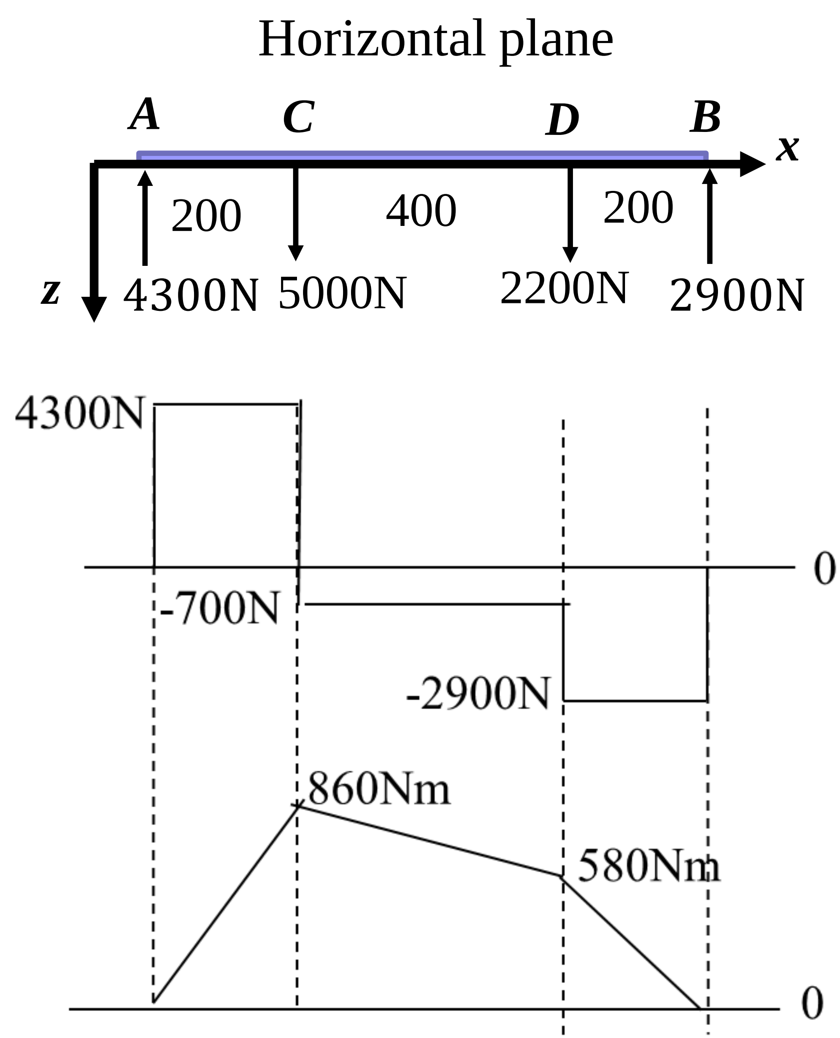
❖ Integrate to get bending moment

$M(x) = 4300\langle x\rangle^{1} - 5000\langle x-0.2\rangle^{1} - 2200\langle x-0.6\rangle^{1} + 2900\langle x-0.8\rangle^{1}$

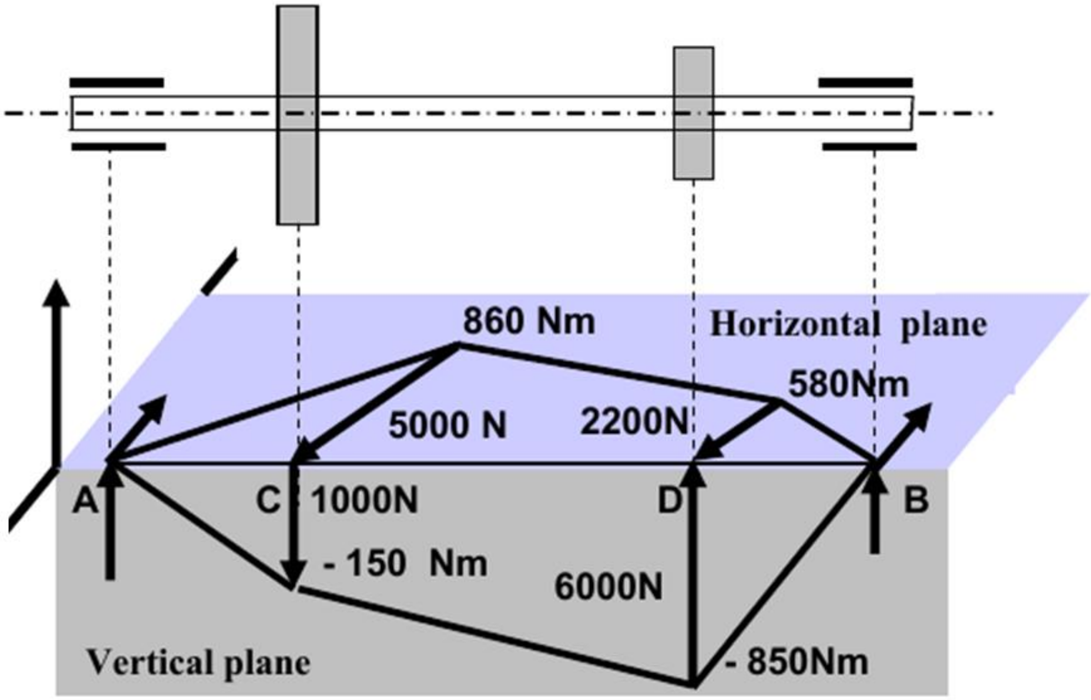
The shear force and bending moment diagrams can be determined

Example 1


Vertical plane
y
x
A
C
D
B
200
400
200
750N
1000N
6000N
4250N
4250N
0
-750N
-1750N
0
-150Nm
-850Nm


Horizontal plane
x
z
A
C
D
B
200
400
200
4300N
5000N
2200N
2900N
4300N
0
-700N
-2900N
860Nm
580Nm
0

Example 1



Resultant bending moment at C

$M_c = \sqrt{150^2 + 860^2} = 873 \text{ Nm}$

Resultant bending moment at D

$M_D = \sqrt{850^2 + 580^2} = 1029 \text{ Nm}$

Section D is critical with bending moment $M_D = 1029$ Nm and $T = 600$ Nm

Example 1

- Design based on $M_D = 1029$ Nm and $T = 600$ Nm
- ASME Code for commercial steel shafting
- for shaft without keyway $\tau_{allow} = 55$ MPa
- for shaft with keyway $\tau_{allow} = 40$ MPa
- Design to cater for keyway and hence select $\tau_{allow} = 40$ MPa Using ASME code with $\mathrm{C_{bm}} = 2$ and $\mathrm{C_t} = 1.5$ with $B = 1$ for solid shaft:

$$\tau_{allow} = \frac{16B}{\pi d_o^3}\left\{\left(\mathrm{C_{bm}}M + \frac{\mathrm{C_{ca}}F d_o(1+\alpha)}{8}\right)^2 + \mathrm{C_t}T^2\right\}^{1/2}$$

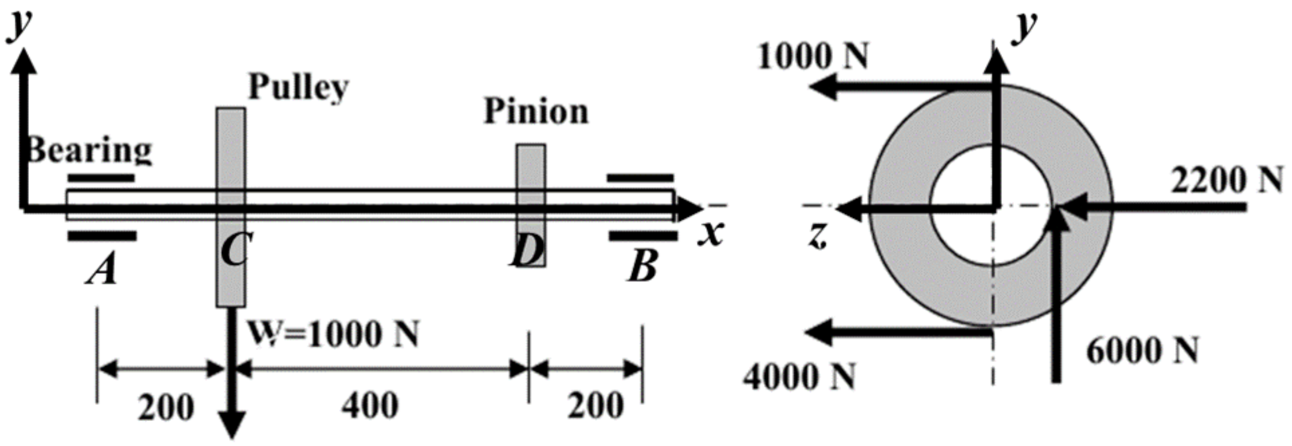
Shaft diameter is

$$d_o^3 = \frac{16B}{\pi \tau_{allow}}\{(\mathrm{C_{bm}}M_D)^2 + \mathrm{C_t}T^2\}^{1/2}$$

Solve to get $d_o = 65.88$ mm; Closest available standard size, select 66 mm

Example 2

Redesign the shaft in example 1 based on NASA ref. using SAE 1006 (HR) steel with yield strength of $\sigma_y = 170$ MPa and factor of safety 2, 3, 4 and 5 (all dimensions in mm)



- Note: Torque is $T = (4000 - 1000)0.2 = 600$ Nm
- Maximum bending moment at D is $M_D = 1029$ Nm
- Material is considered as ductile

Example 2

NASA ref. for ductile materials: safe shaft diameter can be found using

$$d_o^3 = \left(\frac{\text{FS}}{\sigma_y}\right)\frac{32B}{\pi}\left\{\left(M + \frac{Fd_o(1+\alpha)}{8}\right)^2 + \frac{3}{4}T^2\right\}^{1/2}$$

- Given FS = 2; $\sigma_y = 170$ MPa; solid shaft $B = 1,\ \ M = 1029$ Nm; $T =$ 600 Nm with no axial loading. Solve to get $d_o\ = 51.7$ mm
- Similarly for FS = 3, we get $d_o\ = 59.2$ mm
- For FS = 4, we get $d_o\ = 65.1$ mm
- Finally for FS = 5, we get $d_o\ = 70.2$ mm

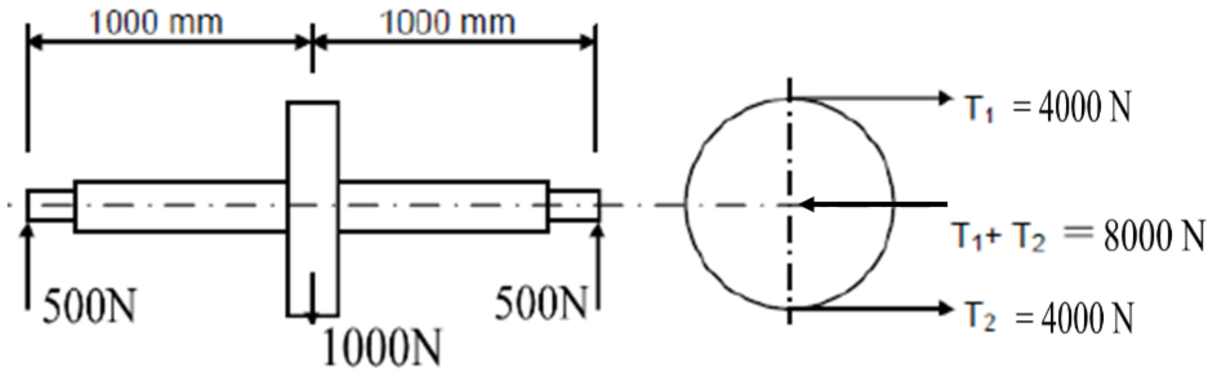
Note: result in example 1 is $d_o\ = 66$ mm

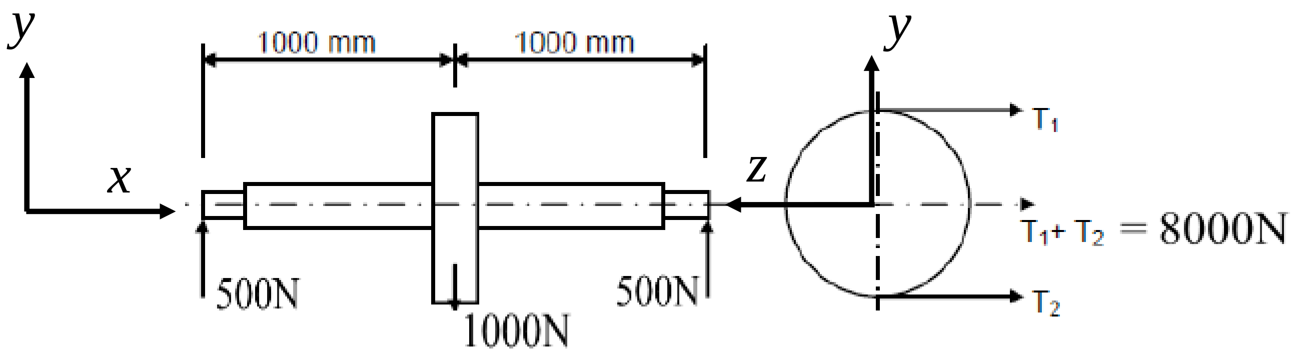
Compare with the ASME code results, the ASME code is equivalent to the NASA ref with a factor of safety of about 4

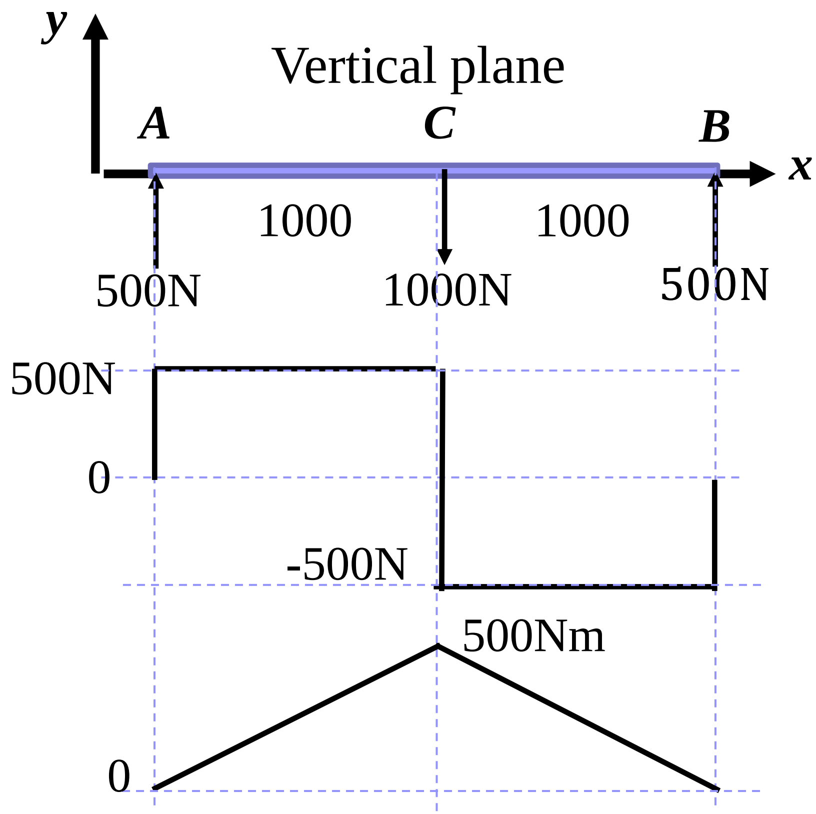
Example 3

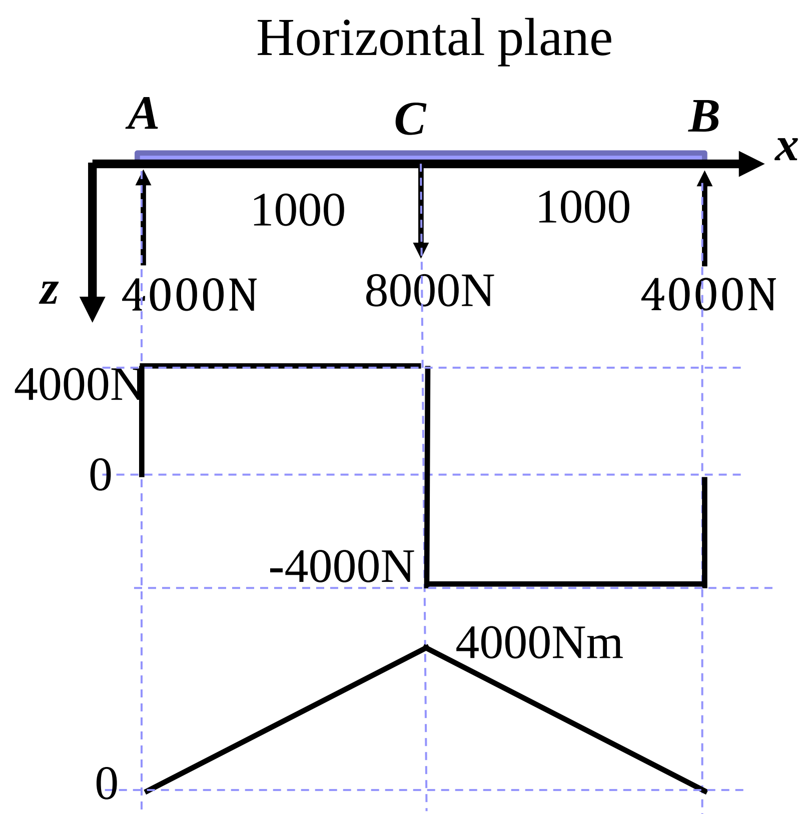
A shaft carries a 1000 N pulley in the middle of two ball bearings which are 2000 mm apart. The pulley is keyed to the shaft and receives 30 kW of power at 150 rpm. The power is transmitted from the shaft through a flexible coupling just outside the right bearing. The belt derive is horizontal and the sum of the belt tension is 8000 N. Calculate the diameter of the shaft based on

a) Equivalent bending moment if the permissible stress in bending of 80 MPa;
b) Equivalent toque if the permissible shear is 45 MPa;
c) Old ASME code with $C_{bm} = 2$ and $C_t = 1.5$




y
x
1000 mm
1000 mm
500N
1000N
500N
y
z
T1
T1+ T2 = 8000N
T2


Vertical plane
y
A
C
B
x
1000
1000
500N
1000N
500N
500N
0
-500N
500Nm
0


Horizontal plane
A
C
B
x
1000
1000
z
4000N
8000N
4000N
4000N
0
-4000N
4000Nm
0

Example 3

Section C is critical with resultant bending moment

$M = \sqrt{500^2 + 4000^2} = 4031$ Nm

Given power 30 kW at 150 rpm; Torque $T\ (\text{Nm}) = 9550\,\frac{P_o\ (\text{kW})}{\eta\ (\text{rpm})} = 1920\ \text{Nm}$

a) For equivalent bending moment if the permissible stress in bending of 80 MPa (note B = 1 for solid shaft) :

$$\sigma_{allow} = \frac{B}{\pi d_o^3}(32M_e)$$

Where $M_e = \frac{1}{2}\left(M + \sqrt{M^2 + T^2}\right) = \frac{1}{2}\left(4031 + \sqrt{4031^2 + 1920^2}\right) = 4250\ \text{Nm}$

$$d_o^3 = \frac{B}{\pi\sigma_{allow}}(32M_e)$$

Solve to get 81.5 mm

Example 3

Found $M = 4031$ Nm

Torque T (Nm) $= 1920$ Nm

b) For equivalent toque if the permissible shear is 45 MPa; (note B = 1 for solid shaft) and for shaft with keyway $\tau_{allow} = 40$ MPa :

$$\tau_{allow} = \frac{16T_e}{\pi d_o^3} B$$

Where $T_e = \sqrt{M^2 + T^2} = \sqrt{4031^2 + 1920^2} = 4460$ Nm

$$d_o^3 = \frac{B}{\pi \tau_{allow}} (16T_e)$$

Solve to get 79.6 mm

Example 3

Found $M = 4031$ Nm

Torque T (Nm) = 1920 Nm

c) For old ASME code with $\mathrm{C_{bm}} = 2$ and $\mathrm{C_t} = 1.5$ (note B = 1 for solid shaft) and for shaft with keyway $\tau_{allow} = 40$ MPa:

$$\tau_{allow} = \frac{16B}{\pi d_o^3}\left\{\left(\mathrm{C_{bm}}M + \frac{\mathrm{C_{ca}}Fd_o(1+\alpha)}{8}\right)^2 + \mathrm{C_t}T^2\right\}^{1/2}$$

Shaft diameter is

$$d_o^3 = \frac{16B}{\pi\tau_{allow}}\{(\mathrm{C_{bm}}M_D)^2 + \mathrm{C_t}T^2\}^{1/2}$$

Solve to get 92.2 mm