MEMS1028
Mechanical Design 1

Lecture 3 part 1

Load & stress analysis (Torsion & Combined loadings)

Objectives

- Analyze torque and torsional shear in power transmission bars, non-circular shafts, closed thin-walled tubes and open thin-walled sections
- Analyze principal stresses and maximum shear stresses in plane stress problems
- Apply generalized 3D stresses and Hooke’s law in engineering design

Power transmission

Shaft power transmission: $P = T\omega$

- ❖ Angular frequency: $\omega = 2\pi f$ (rad/s)
- ❖ Frequency f in Hz (rev/s)
- ❖ Power P (Watts)
- ❖ Torque T (Nm)

Solid shaft: polar moment of inertia

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{2} r^4$$

Hollow shaft; polar moment of inertia

$$J = \frac{\pi}{32}\left(d_o^4 - d_i^4\right) = \frac{\pi}{2}\left(r_o^4 - r_i^4\right)$$

Torque and torsion

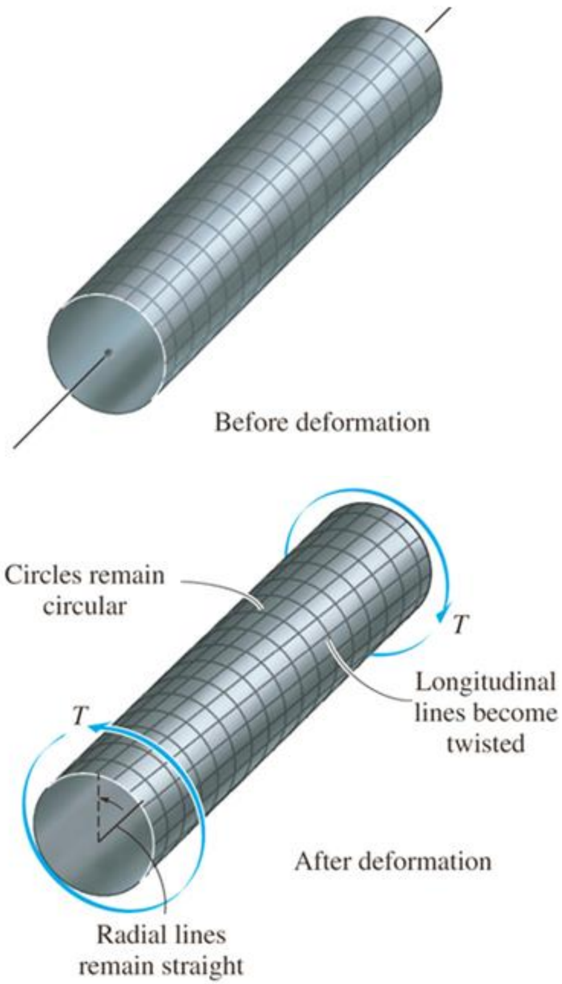
For circular member:

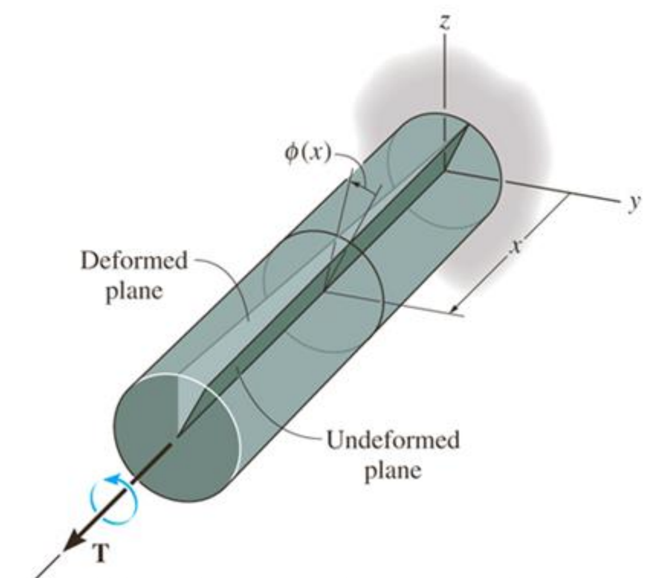
Shear stress $\tau = \frac{T\rho}{J}$ $\qquad$ $\tau_{max} = \frac{Tr}{J}$

Shear strain $\gamma = \frac{\rho\phi}{J}$ $\qquad$ Angle of twist $\phi = \frac{TL}{JG}$

$\phi \propto T$

$\phi \propto L$



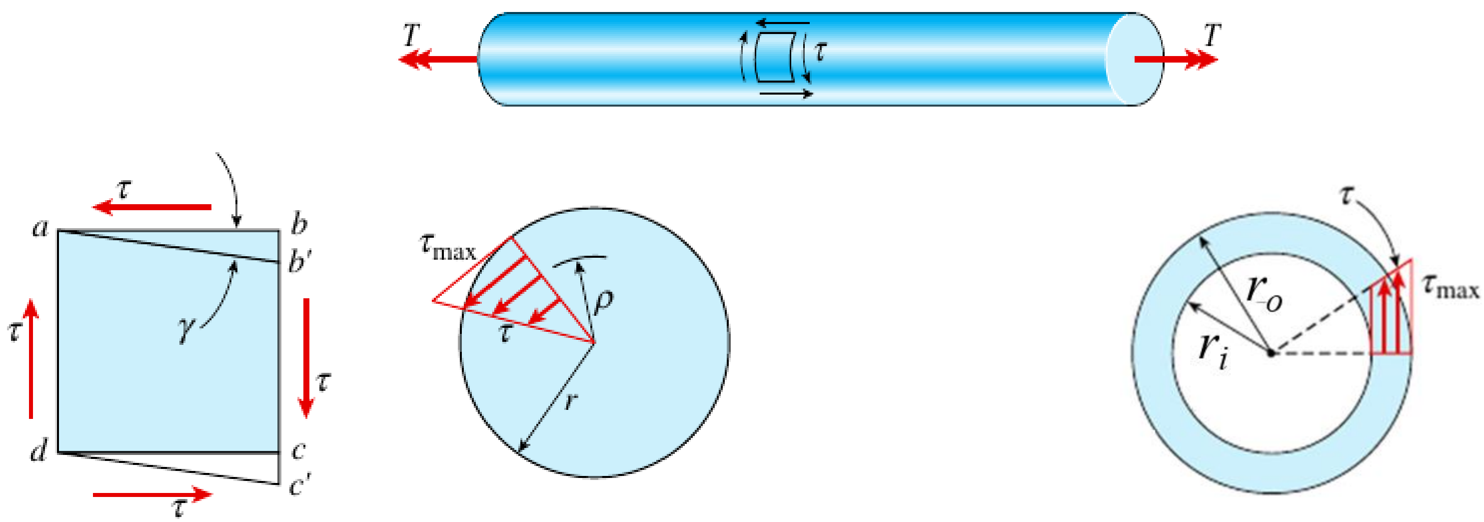


The angle of twist $\phi(x)$ increases as x increases.

If linear elastic, Hooke's law applies:
$\tau = G\gamma$

Circular shafts

Stress profiles:



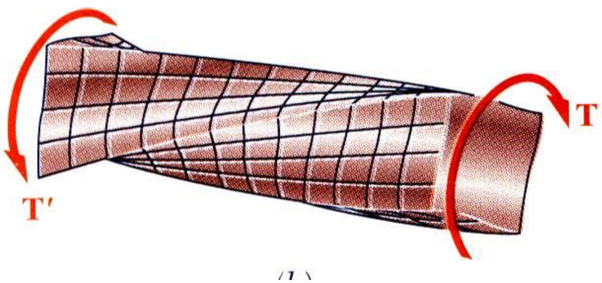
Solid shaft:

$$\frac{J}{r} = \frac{\pi}{2} r^3 = \frac{T}{\tau_{max}}$$

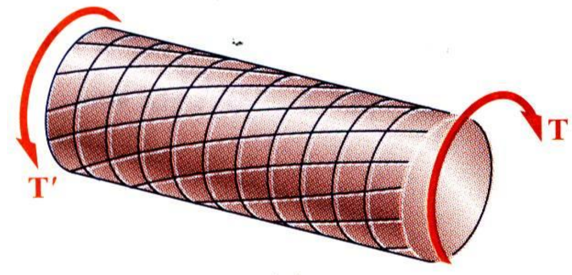
Hollow shaft:

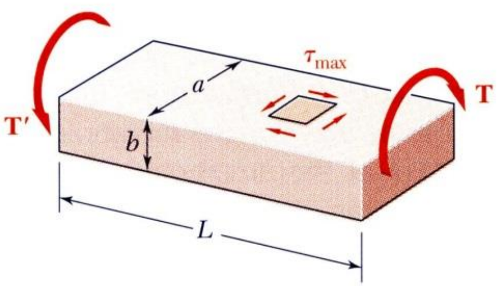
$$\frac{J}{r_o} = \frac{\pi}{2} \frac{(r_o^4 - r_i^4)}{r_o} r^3 = \frac{T}{\tau_{max}}$$

Non-circular shafts



- Cross-sections of noncircular (non-axis-symmetric) shafts are distorted when subjected to torsion





$$\tau_{max} = \frac{T}{c_1 a b^2}$$

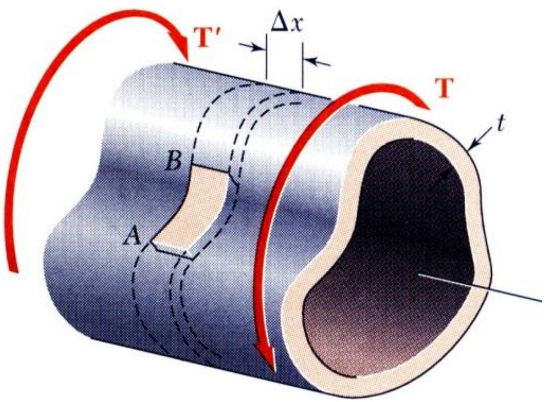
$$\phi = \frac{TL}{c_2 a b^3 G}$$

TABLE Coefficients for Rectangular Bars in Torsion

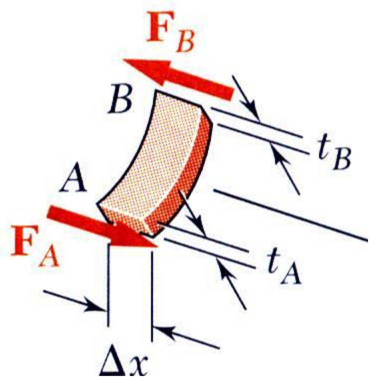
a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Shape of cross-section	τ_{max}	ϕ
Square	$\frac{4.81\,T}{a^3}$	$\frac{7.10\,TL}{a^4 G}$
Equilateral triangle	$\frac{20\,T}{a^3}$	$\frac{46\,TL}{a^4 G}$
Ellipse	$\frac{2\,T}{\pi a b^2}$	$\frac{(a^2+b^2)TL}{\pi a^3 b^3 G}$

Closed thin-walled tubes



- For closed thin-wall tubes (i.e. $r >> t$) under torsion; summing forces in the x-direction on AB:

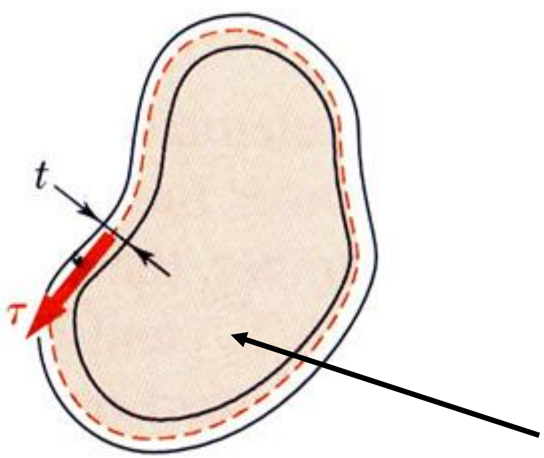


Shear flow is constant

$$\tau_A t_A = \tau_B t_B = \tau_{ave} t = q \text{ (shear flow)}$$

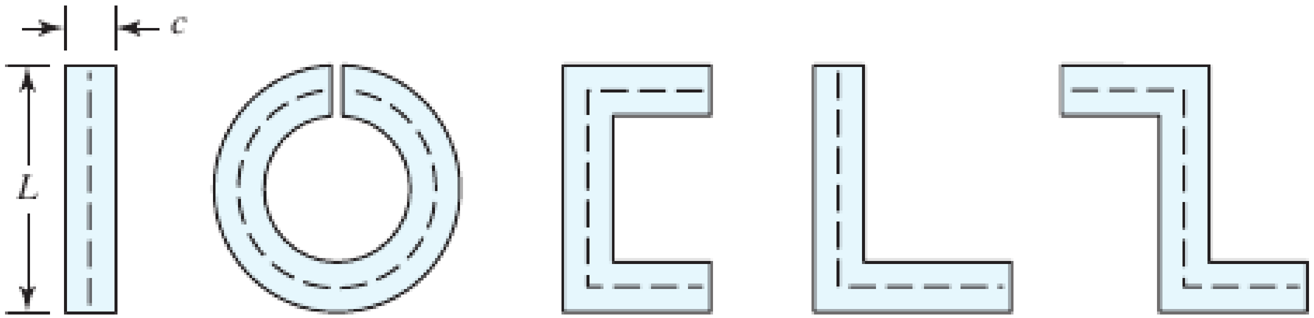
For constant wall thickness t:

$$\tau_{ave} = \frac{T}{2tA_m} \qquad \theta_1 = \frac{TL_m}{4GA_m^2 t}$$



θ_1 = angular twist (rad) per unit of length of the tube;
Torque T (Nm); G = shear modulus;
A_m = area enclosed by the section median line;
L_m = length of section median line

Open thin-walled sections

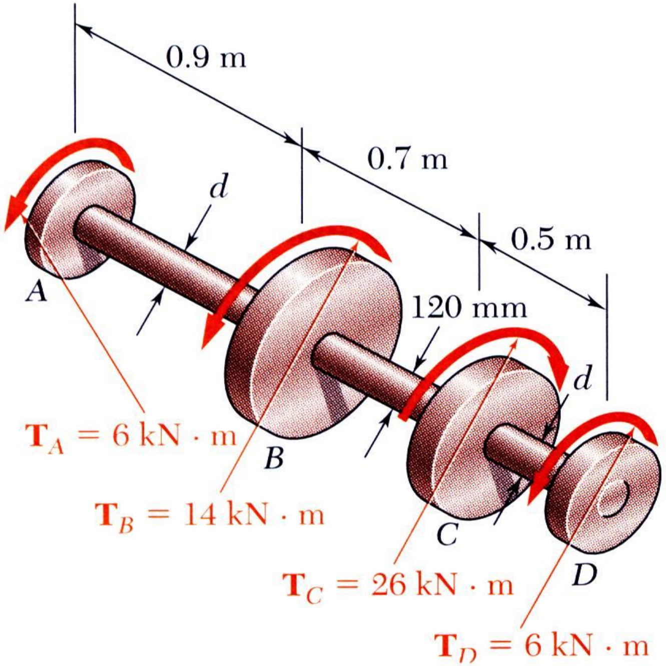


For open thin-walled sections under torsion:

$$\tau = G\theta_1 c = \frac{3T}{Lc^2}$$

- ❖ θ_1 = angular twist (rad) per unit of length of the tube;
- ❖ Torque T (Nm); G = shear modulus;
- ❖ Wall thickness = c;
- ❖ L = length of median line

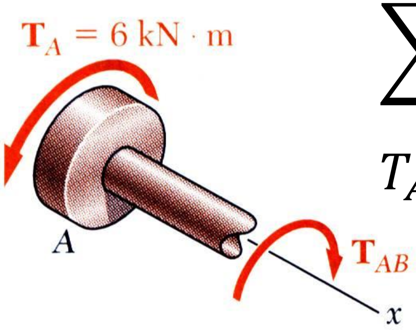
Illustrative example 1



Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid of diameter *d*. For the loading shown, determine the required diameter *d* of shaft *AB* if the allowable shearing stress in these shafts is 65 MPa.

Illustrative example 1

Cut sections through shafts *AB* and *BC* and perform static equilibrium analysis to find torque loadings



$$\sum M_x = 0 = (6) - T_{AB}$$

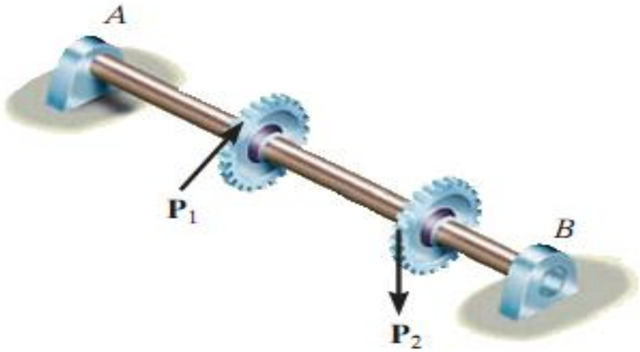
$$T_{AB} = 6\text{kNm}$$

Given τ_{max}=65MPa:

$$\tau_{max} = \frac{Tr}{J} = \frac{T_{AB}r}{\frac{\pi}{2}r^4}$$

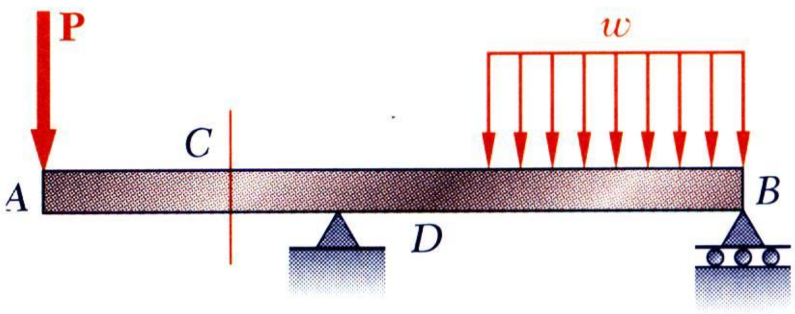
$r = 38.9(10^{-3})$m or d =77.8mm

Combined loadings



- A shaft is normally designed to resist both torsion and bending stress

Transverse loading applied to a short beam results in normal and shearing stresses in transverse sections (which cannot be ignored)

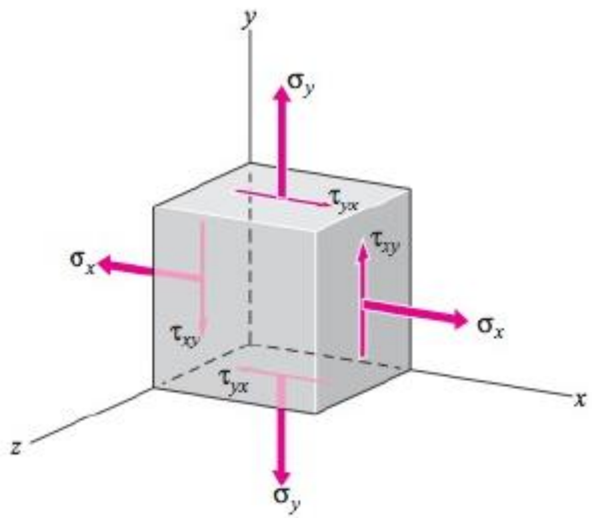


How should we consider these combinations of different stresses?

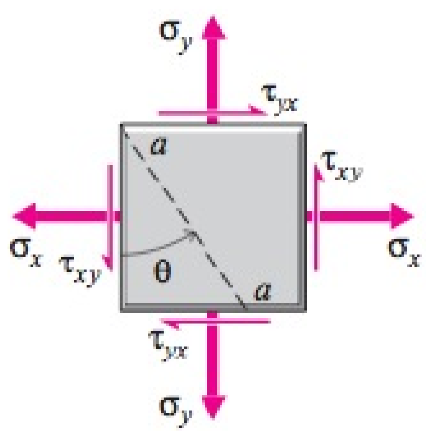
Summary of basic loadings

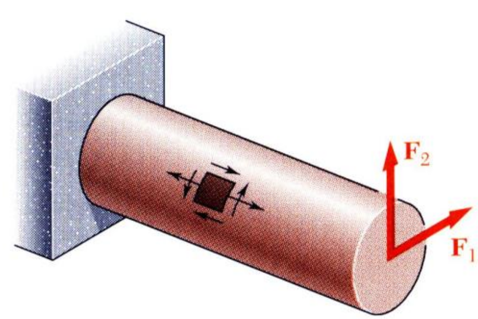
Loading	Normal stress (σ)	Shear stress (τ)
Axial	$\sigma = \frac{P}{A}$	-
Direct shear	-	$\tau = \frac{V}{A}$
Torsion	-	$\tau = \frac{Tr}{J}$
Bending	$\sigma = -\frac{My}{I}$	$\tau = \frac{VQ}{It}$

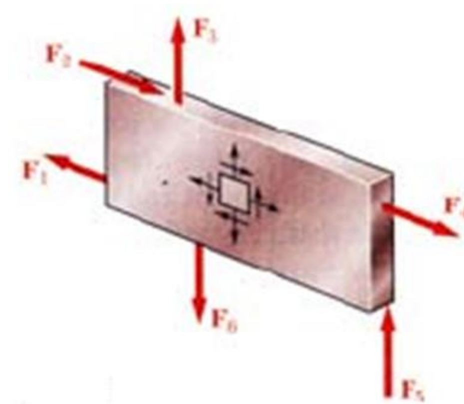
Plane stress



- Many practical engineering problems can be simplified by considering only three independent stresses – called plane stress on a 2D element (i.e. two parallel faces of the small element shown are assumed to be free of stress)
- Example: state of plane stress occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force

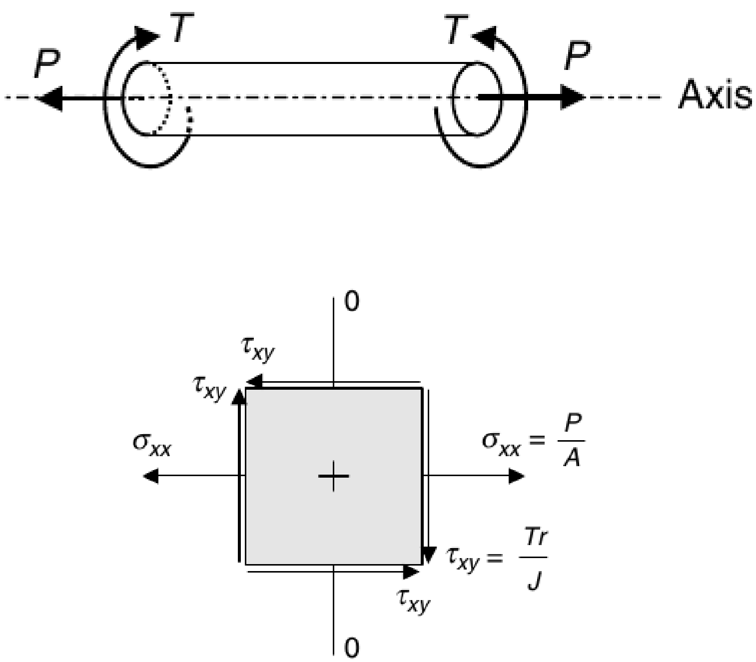




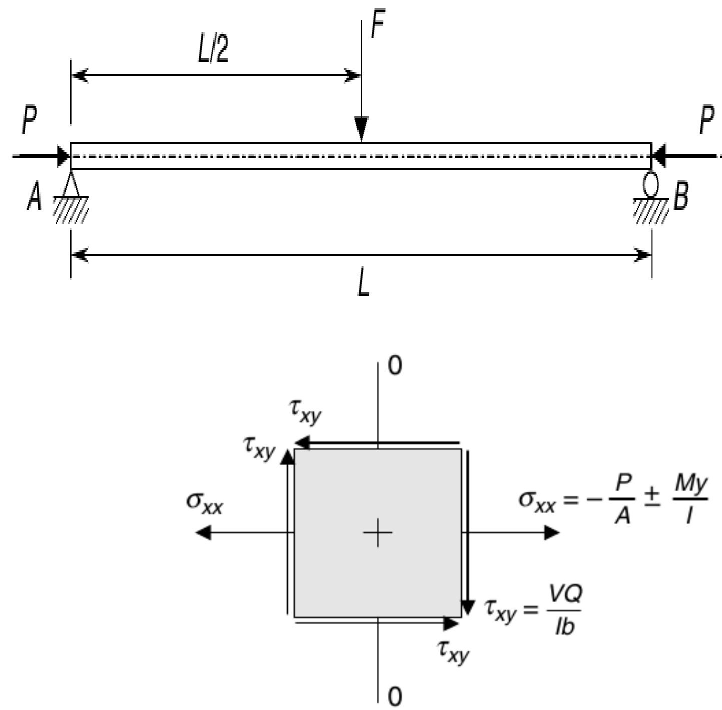


Illustrative combinations

Axial & torsion

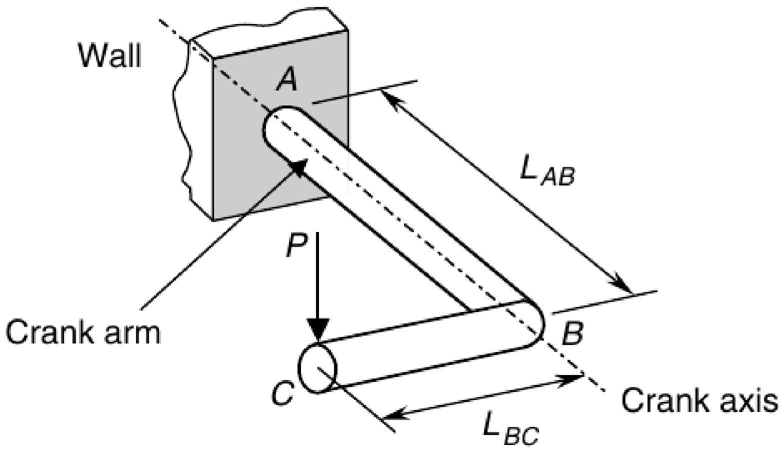


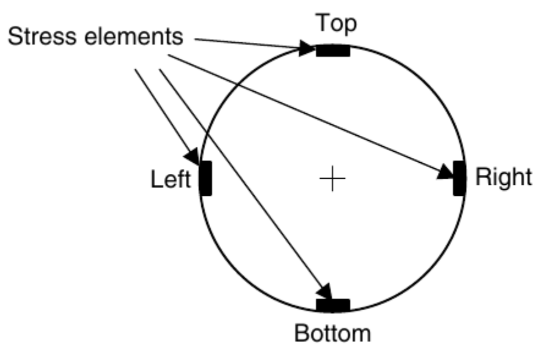
Axial & bending



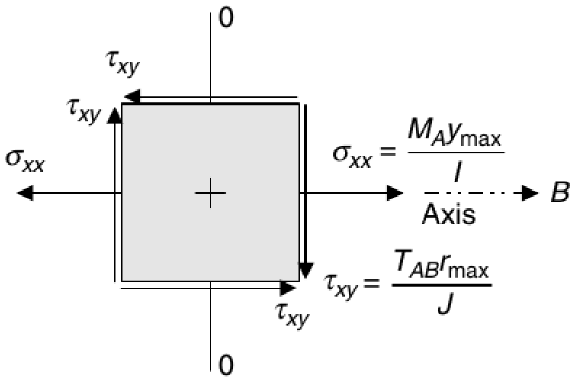
Illustrative combinations

Torsion & bending

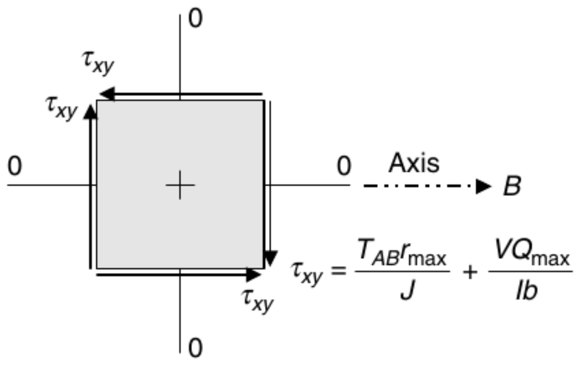




Top element at B:

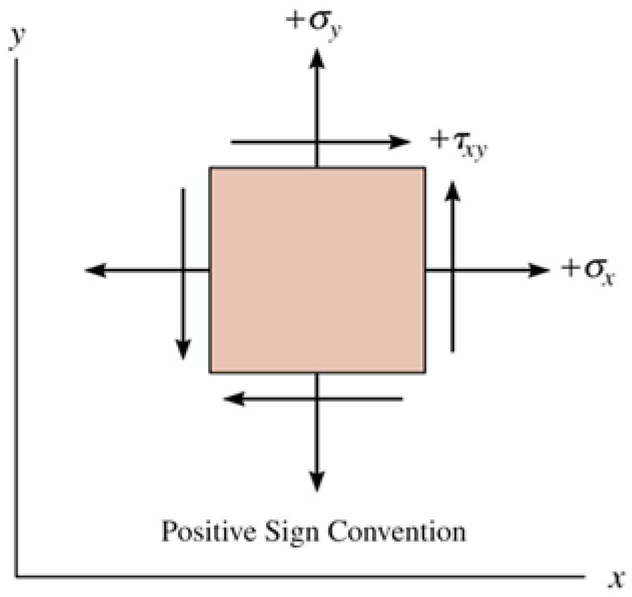


Left element at B:

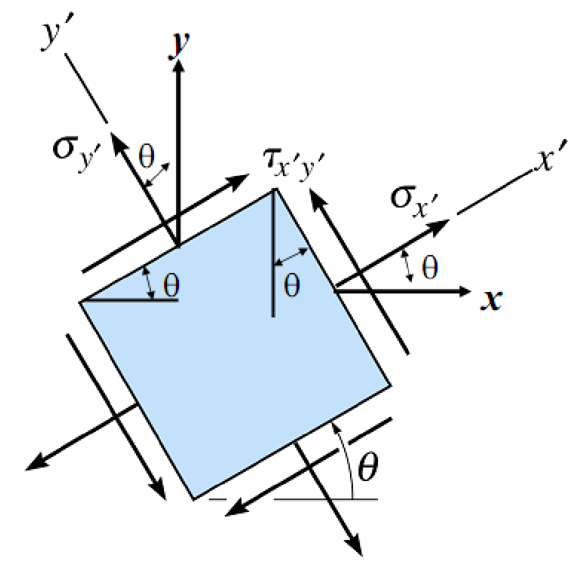


Plane stress transformation

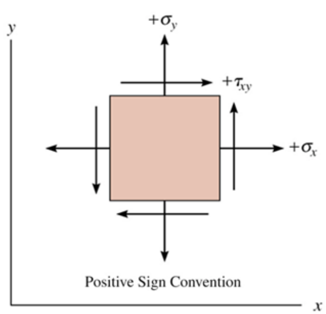
Given Plane Stress State:



What are new stresses after a rotation of θ?



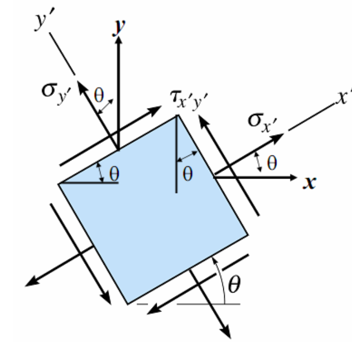
Plane stress transformation



$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



- Tensile normal stresses are positive
- Shearing stress pointing in positive direction of axis defined by second subscript is positive
- Positive angle measured counterclockwise from reference *x*-axis

Principal stresses

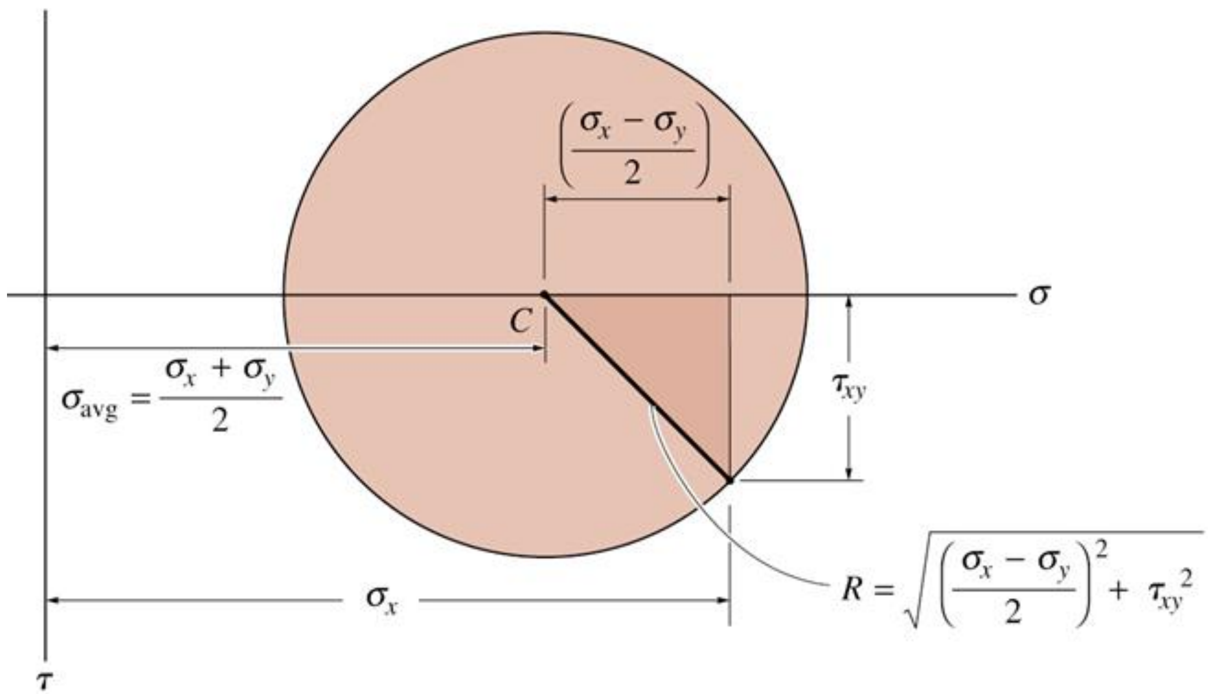
- The transformation equations provides a means for finding the normal and shearing stresses on different planes through a point in the stressed element
- As the element is rotated through an angle θ, the normal and the shearing stresses on different planes vary continuously
- For design purposes, critical stress at the point are usually the maximum tensile (or compressive) and shearing stresses
- The principal stresses are the maximum normal stress σ_{max} and minimum normal stress σ_{min}.
- How do we find the principal stresses and maximum shearing stress?

Mohr's circle – plane stress

- Mohr's circle is a pictorial interpretation of the transformation equations for plane stress. The equations can be put into the follow format:

$$\left[\sigma_{x'} - \sigma_{avg}\right]^2 + \tau_{x'y'}^2 = R^2 \qquad \sigma_{avg} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

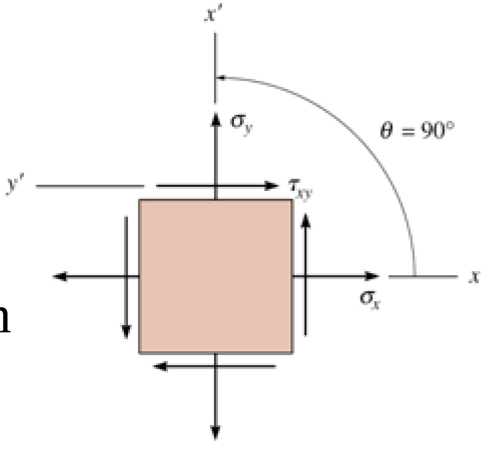
- These equations represent a circle center at (σ_{avg}, 0) with radius R
- Each point on the circle represent the normal and shearing stresses on one plane through the stressed point



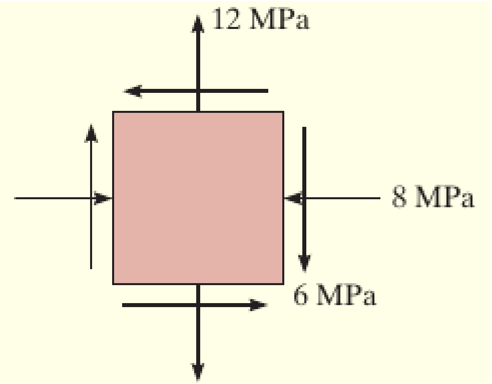
Procedure for analysis

1) Choose a set of x-y coordinate axes
2) Identify the stresses σ_x, σ_y and $\tau_{xy} = -\tau_{yx}$ with proper sign
3) Draw a set of $\sigma\tau$-coordinate axes with σ positive to the right and τ upward as positive
4) Plot the point (σ_x, τ_{xy})
5) Plot the point (σ_y, τ_{yx})
6) Join the 2 points to locate the center C and the radius R of Mohr's circle
7) Draw the circle
8) Locate the principal stresses and θ_p
9) Locate the maximum shearing stress and θ_s

- ❖ Normal stresses σ are plotted as horizontal coordinates, with tensile stresses (positive) plotted to the right of the origin
- ❖ Shearing stresses τ are plotted as vertical coordinates, with those tending to produce a clockwise rotation of the stress element plotted above the origin



Illustrative example 2

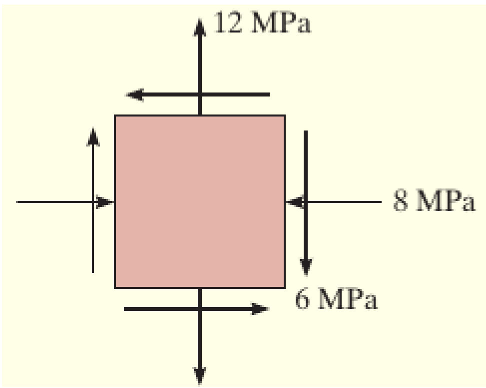


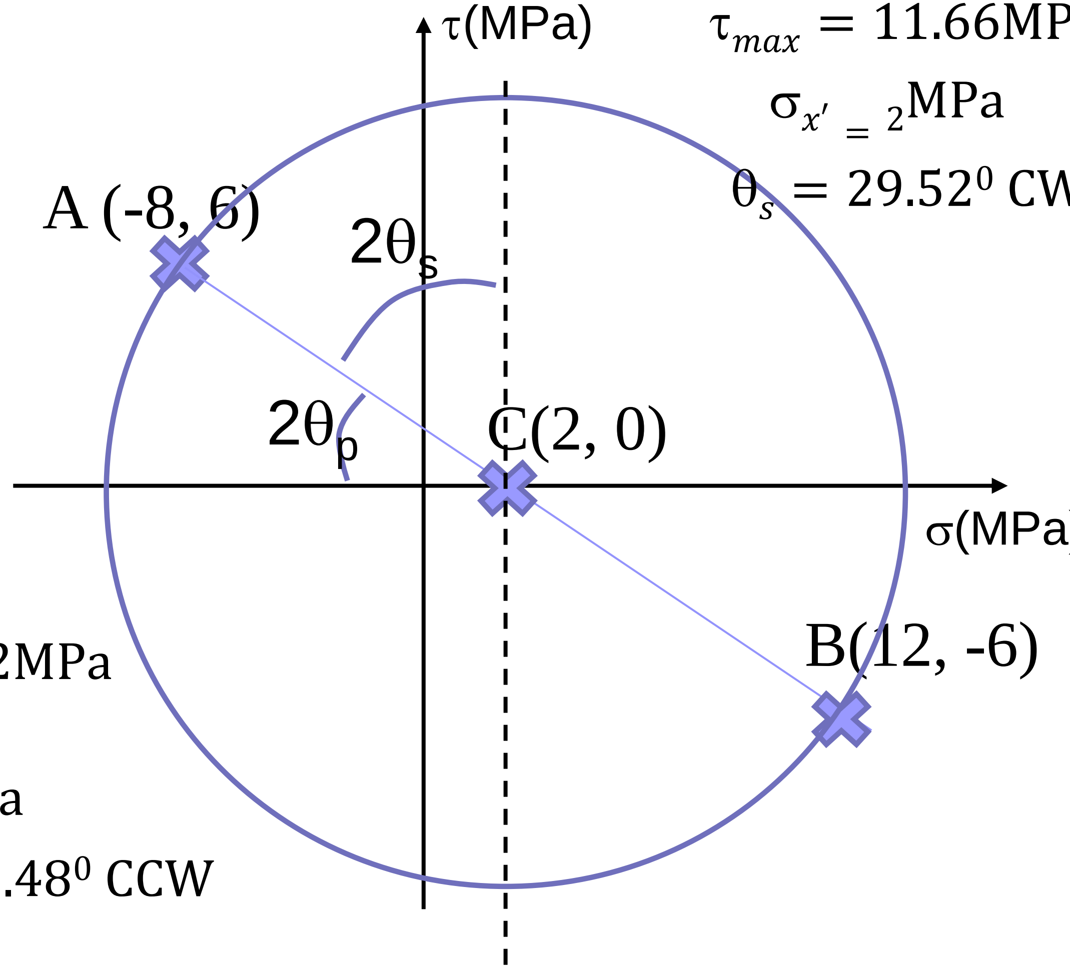
For the state of plane stress shown, use Mohr's circle to determine the principal planes, principal stresses, the maximum in plane shearing stress and the corresponding normal stress.

Determine the stress components exerted on the element obtained by rotating the given element counterclockwise through 30 degrees.

σ_x = -8MPa
τ_{xy} = 6MPa
σ_y = 12MPa
and τ_{yx} = -6MPa
(σ_x, τ_{xy}) = (-8, 6) MPa
(σ_y, τ_{yx}) = (12, -6) MPa

Illustrative example 2





$\tau_{max} = 11.66$MPa

$\sigma_{x'} = 2$MPa

$\theta_s = 29.52^0$ CW

A=(σ_x, τ_{xy}) = (-8, 6) MPa

B=(σ_y, τ_{yx}) = (12, -6) MPa

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{-8 + 12}{2} = 2\text{MPa}$$

$$R = \sqrt{(10)^2 + 6^2} = 11.66\text{MPa}$$

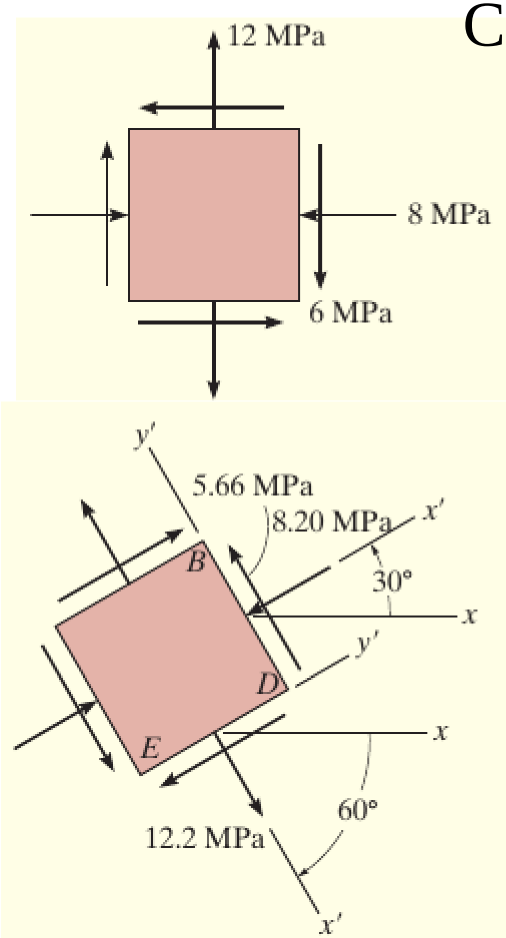
$\sigma_1 = 13.66$MPa

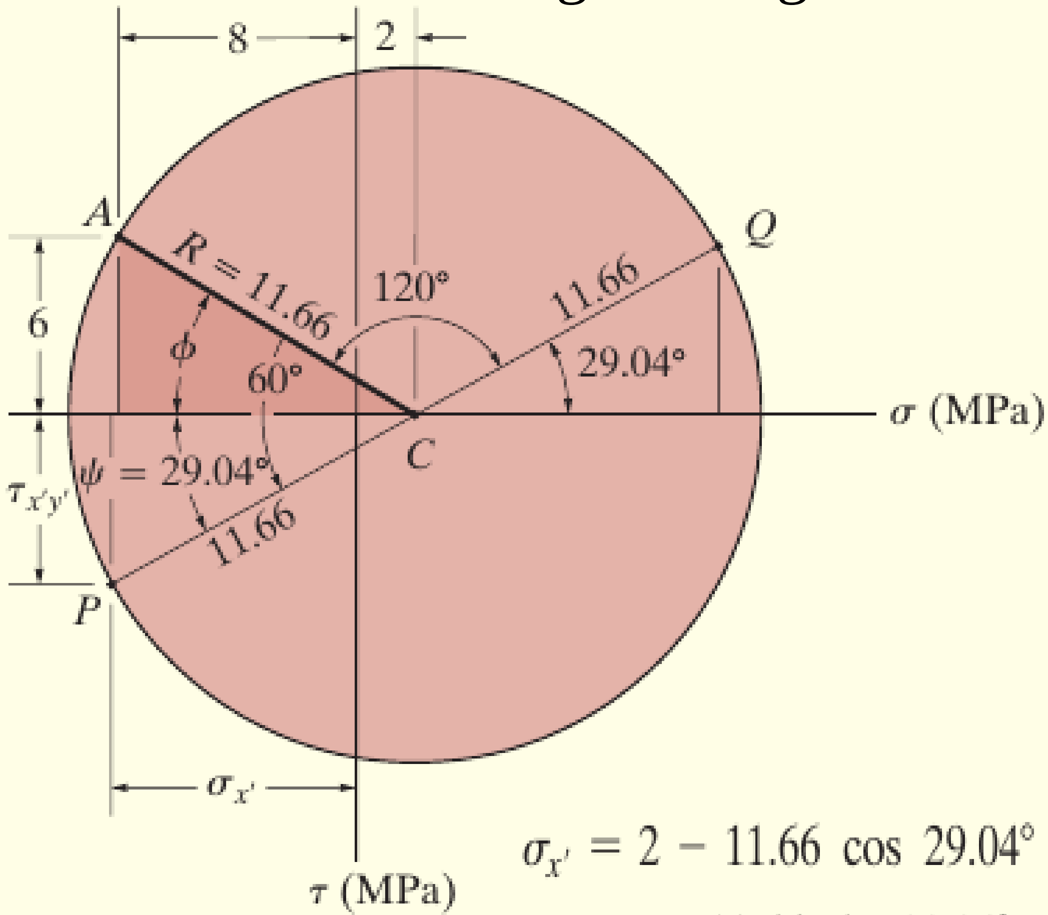
$\theta_p = 15.48^0$ CCW

$\sigma_2 = -9.66$MPa

Illustrative example 2

Counterclockwise through 30 degrees or $2\theta = 60^0$





$$\sigma_{x'} = 2 - 11.66 \cos 29.04^\circ = -8.20 \text{ MPa}$$

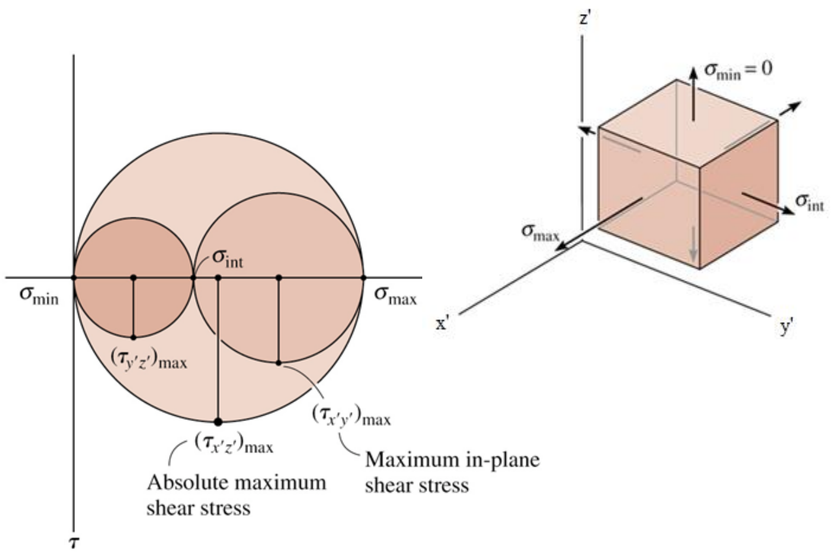
$$\tau_{x'y'} = 11.66 \sin 29.04^\circ = 5.66 \text{ MPa}$$

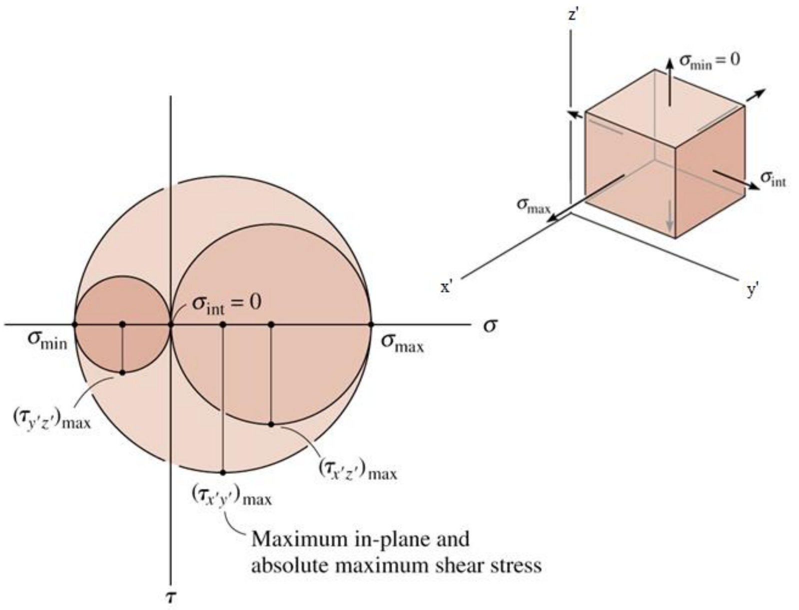
Maximum shear stress

Consider the 3D element with three principal stresses $\sigma_{max} \geq \sigma_{int} \geq \sigma_{min}$ (with one of them being 0)

- Case 1: σ_{min} =0
- Case 2: σ_{int} =0

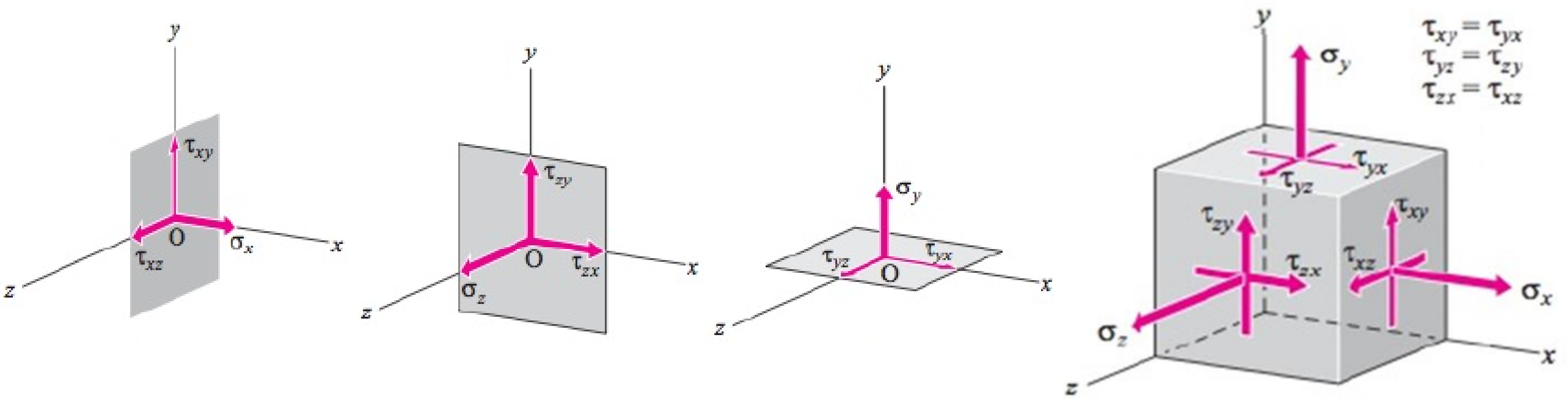
Viewing the element in 2D (i.e. in the y'z', x'z', and x'y' planes) we can use Mohr's circle to determine the maximum shear stress for each case:





General stress at a point

For a 3D element at a point, the specification of stresses on three mutually perpendicular planes is sufficient to completely describe the state of stresses



Shearing stresses τ has 2 subscripts, the first subscript designates the normal to the plane on which the stress acts and the second designate the coordinate axis to which the stress is parallel.

General 3D stresses

In plane stress, the components of the state of stress depend on the orientation of the coordinate system in which they are expressed. Suppose that we know the components of stress at a point "*p*" in terms of a particular coordinate system *xyz*:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

These components will generally have different values when expressed in terms of a coordinate system *x'y'z'* having a different orientation. For any state of stress, at least one coordinate system *x'y'z'* exists for which the state of stress is of the form

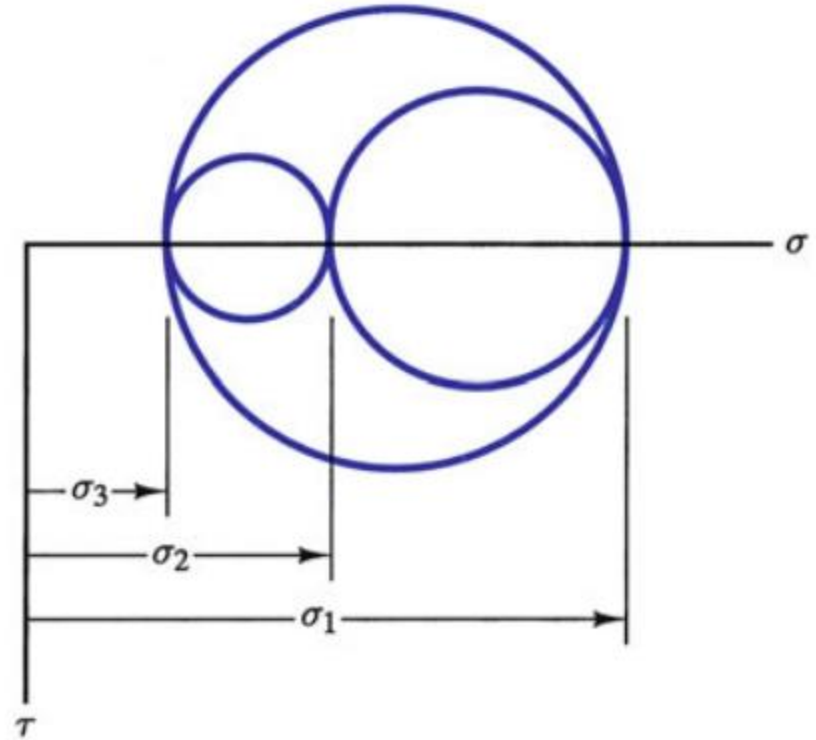
$$\begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{y'x'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{z'x'} & \tau_{z'y'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The axes *x'*, *y'*, *z'* are called principal axes and σ_1, σ_2, and σ_3 are the principal stresses

General 3D stresses

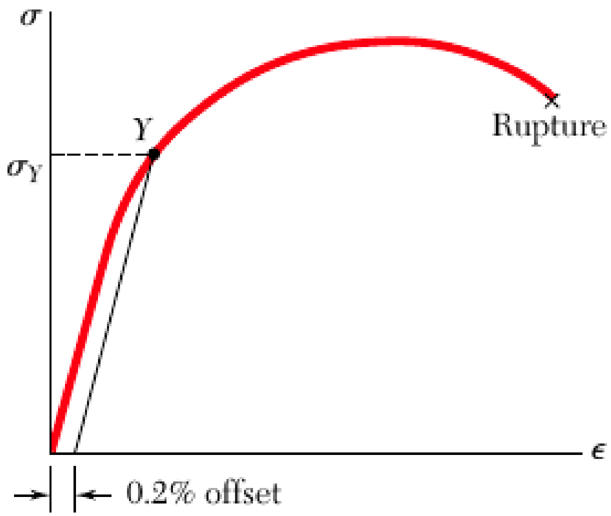
- The state of stress can be visualized by superimposing the Mohr's circles
- Notice that if $\sigma_1 > \sigma_2 > \sigma_3$, the absolute maximum shear stress is

$$|\tau_{max}| = \frac{\sigma_1 - \sigma_3}{2}$$



Hooke's law

- Assume the material is uniform throughout the body (i.e. homogeneous); has the same properties in all directions (i.e. isotropic material); and linearly elastic
- When loading within elastic limit, Hooke's law states that stress is proportional to strain

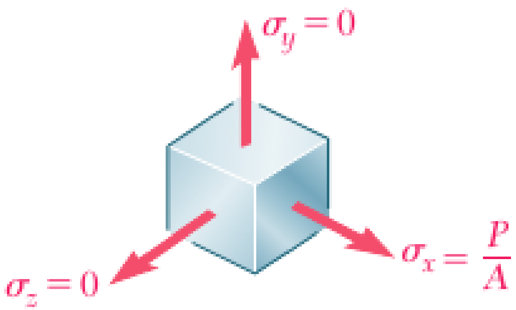


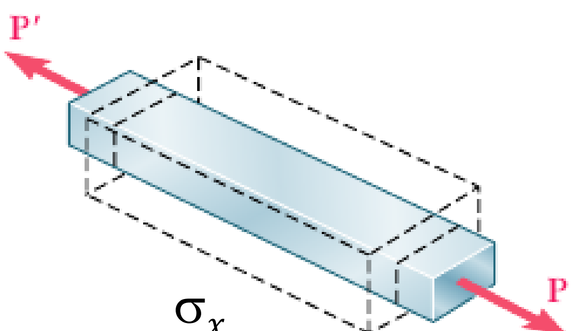
$$\sigma_x = E\varepsilon_x$$

E = Modulus of elasticity (Young's modulus)

How does this apply in 3D?

Poisson's ratio





$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$

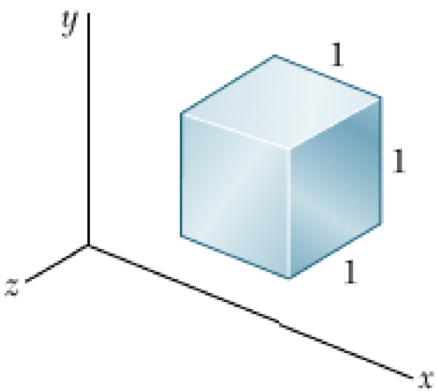
- In all engineering materials, the elongation produced by an axial tensile force *P* in the direction of the force is accompanied by a contraction in any transverse direction
- The relationship between lateral and axial strain is called Poisson's ratio ν

Note: For homogeneous and isotropic material, $\varepsilon_y = \varepsilon_z$

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

ν = Poisson's ratio

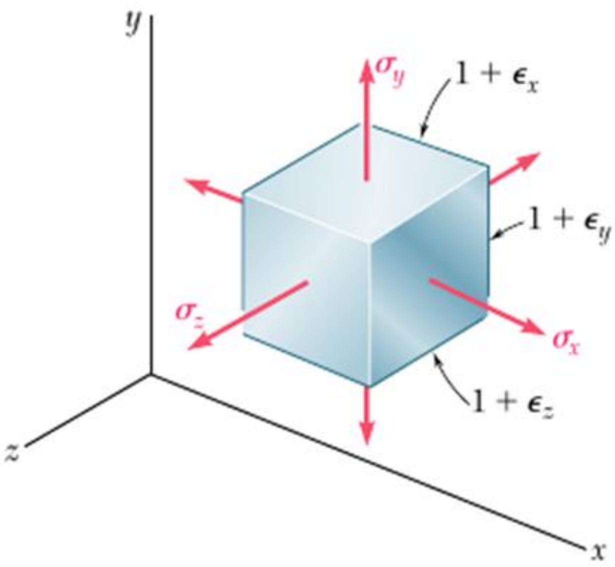
Generalized Hooke's law



$$\varepsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_y = +\frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_z}{E}$$

$$\varepsilon_z = +\frac{\sigma_z}{E} - \frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$



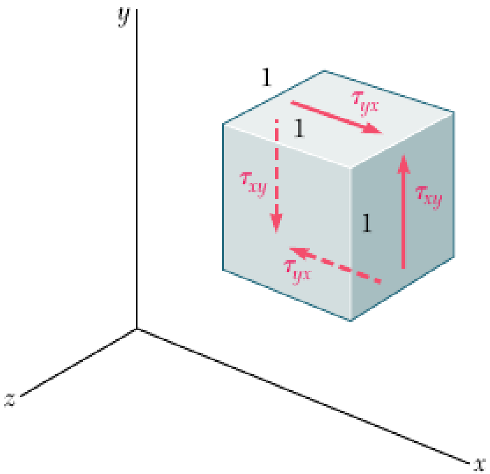
These equations can be rewritten as:

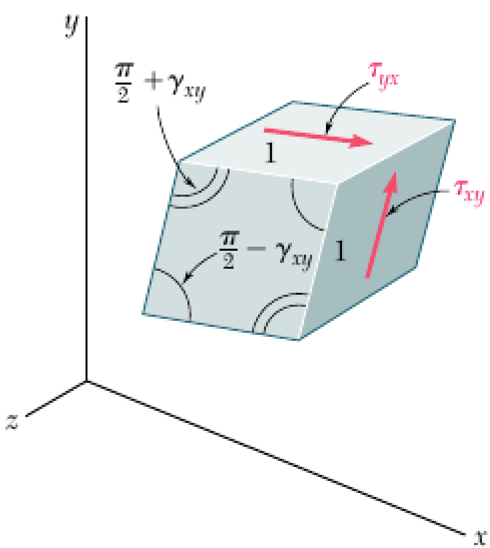
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)]$$

Shearing strain





Hooke's law for elastic shearing strain:

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx}$$

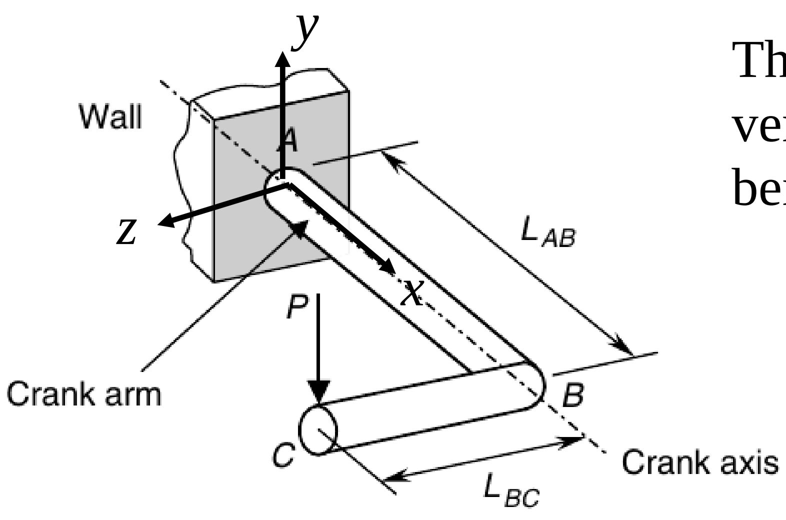
G = Modulus of rigidity (Shear modulus)

Shear modulus (G) is related to Young's modulus (E) by Poisson's ratio (ν):

$$G = \frac{E}{2(1+\nu)}$$

Example 1

Determine the stresses on the top element at B of a solid circular crank arm shown subjected to a downward applied force (P), where P = 2.25kN; L_{AB} = 0.8m; L_{BC} = 0.4m; R = 0.025m and determine the principal stresses



The reactions at the wall will consist of a vertical shear force $V = P$ = 2.25kN (↑); a bending moment (M_A) and a torque (T_{AB})

$$M_A = P(L_{AB}) = -1.8\text{kNm (but along } \bar{k})$$

$$T_{AB} = P(L_{BC}) = 900\text{Nm } (-\bar{\imath})$$

For the top element at B:

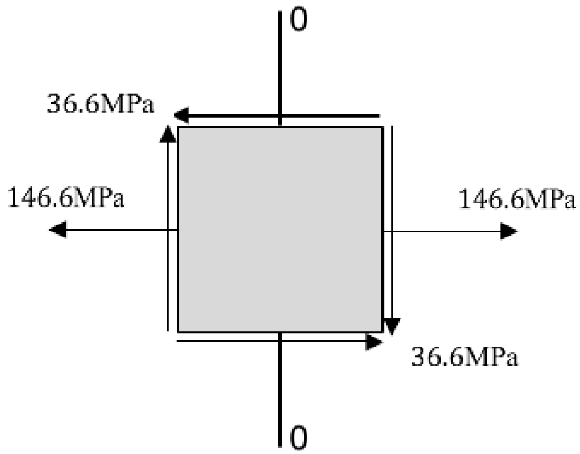
$$I = \frac{1}{4}\pi R^4 = 3.07(10^{-7})\ \text{m}^4;$$

$$J = \frac{1}{2}\pi R^4 = 6.14(10^{-7})\ \text{m}^4;$$

$$\sigma_x = -\frac{M_A R}{I} = 146.6\text{MPa (top is in tension)}$$

$$\tau_{xy} = \frac{T_{AB} R}{J} = 36.6\text{MPa}$$

Example 1



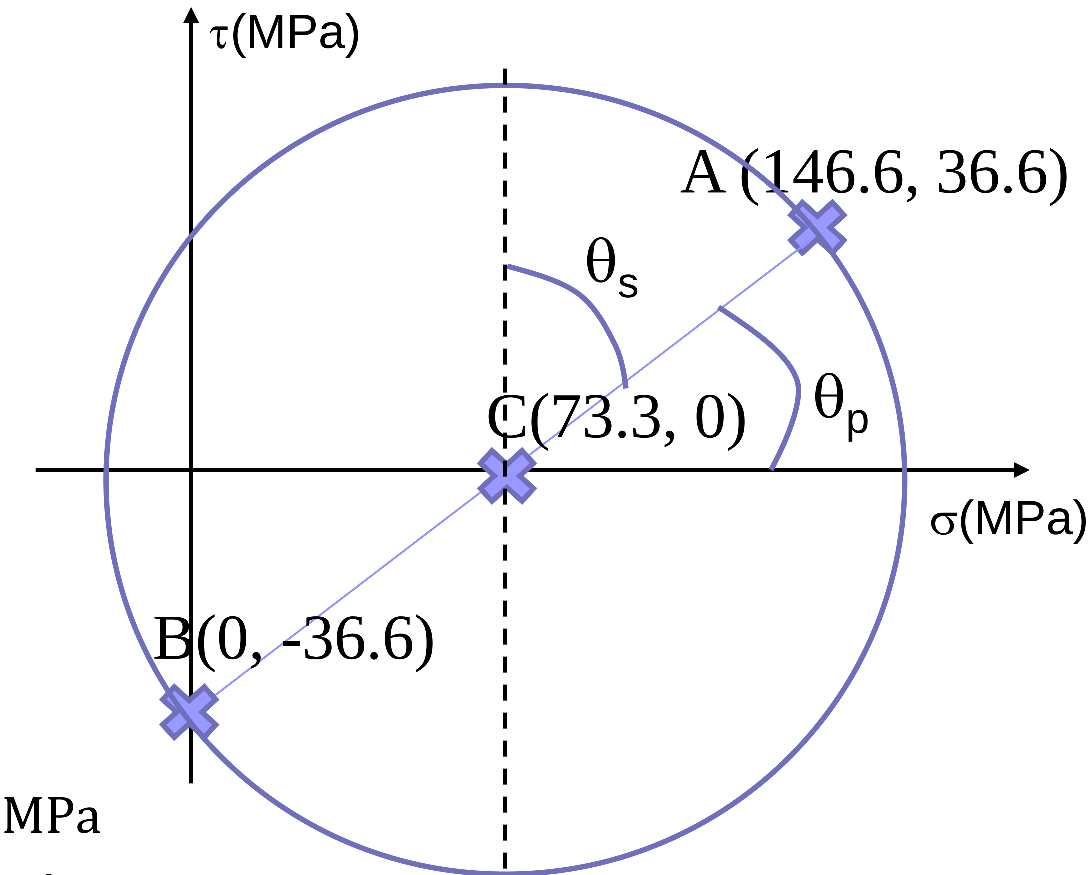
A=(σ_x, τ_{xy}) = (146.6, 36.6) MPa
B=(σ_y, τ_{yx}) = (0, -36.6) MPa

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = 73.3\text{MPa}$$

$$R = \sqrt{(73.3)^2 + 36.6^2} = 81.9\text{MPa}$$

$\sigma_1 = 155.2\text{MPa}$ $\theta_p = 13.3^0$ CW

$\sigma_2 = -8.6\text{MPa}$

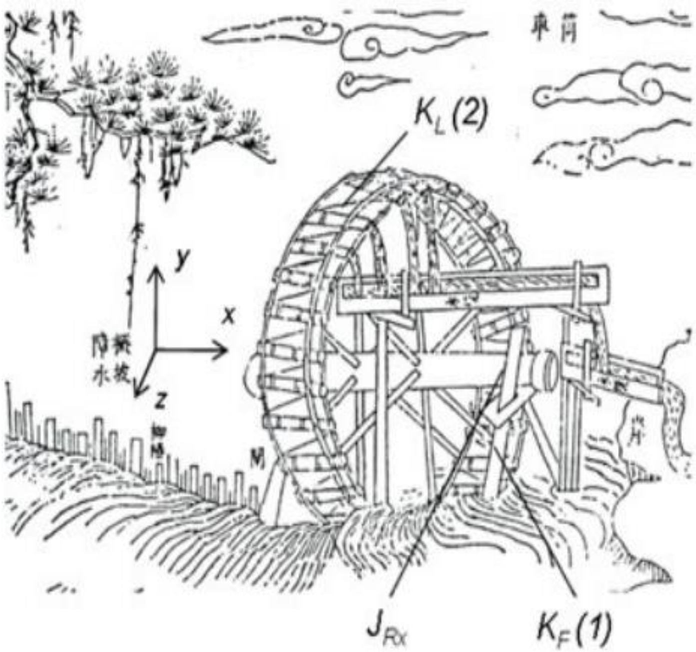


$\tau_{max} = 81.9\text{MPa}$ $\theta_s = 31.7^0$ CCW

$\sigma_{x'} = 73.3$MPa

Ancient Chinese mechanisms

A cylinder wheel Tong Che (筒車)



How would you analyse the stresses in the supporting beam?