MEMS1028 Mechanical Design 1

Lecture 2

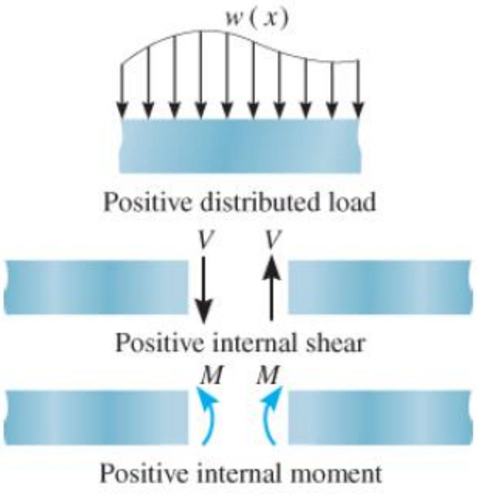
Load & stress analysis (Beams)

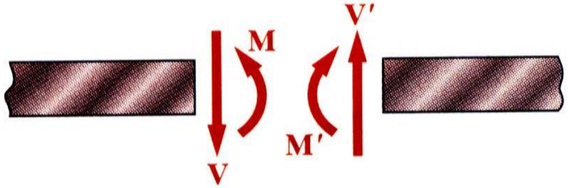
Objectives

- Construct shear force & bending moment diagrams for different beam loadings and support conditions
- Analyze the flexure stress & transverse shear in beams
- Design of prismatic beams and curved beams based on given specifications
- Understand the limitations of the design

Introduction

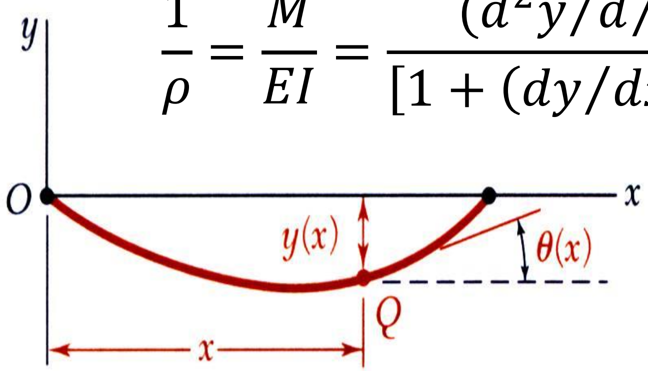
- A beam is a common structural member subjected to transverse loads to withstand significant bending effects as oppose to twisting or axial deformation
- When a beam is loaded by forces or couples, stresses and strains are created throughout the interior of the beam
- To determine these stresses and strains, the internal forces and internal couples that act on the cross sections of the beam must be found





(a) Internal forces
(positive shear and positive bending moment)

Shear force & bending moment



$$\frac{1}{\rho} = \frac{M}{EI} = \frac{(d^2y/d/x^2)}{[1 + (dy/dx)^2]^{3/2}}$$

For small θ: $\frac{1}{\rho} \approx \frac{d^2y}{dx^2}$

$$EI\frac{d^2y}{dx^2} = M(x)$$

$\frac{dM}{dx} = V$ or $V_D - V_C = -\int_{V_C}^{V_D} w\,dx$

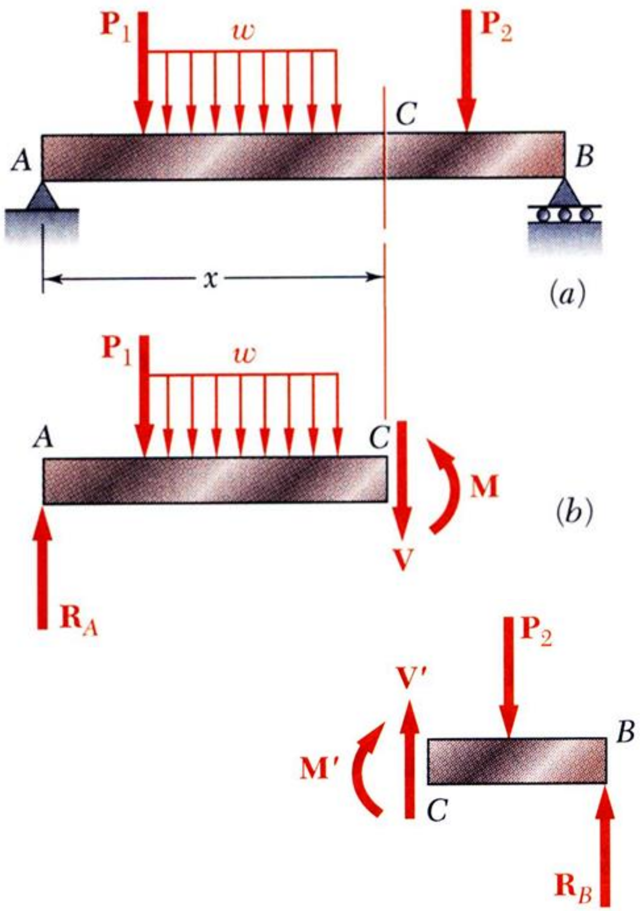
$$\Rightarrow \quad EI\frac{d^3y}{d^3x} = V(x)$$

$\frac{dV}{dx} = -w$ or $M_D - M_C = \int_{x_C}^{x_D} V\,dx$

$$\Rightarrow \quad EI\frac{d^4y}{dx^4} = -w(x)$$

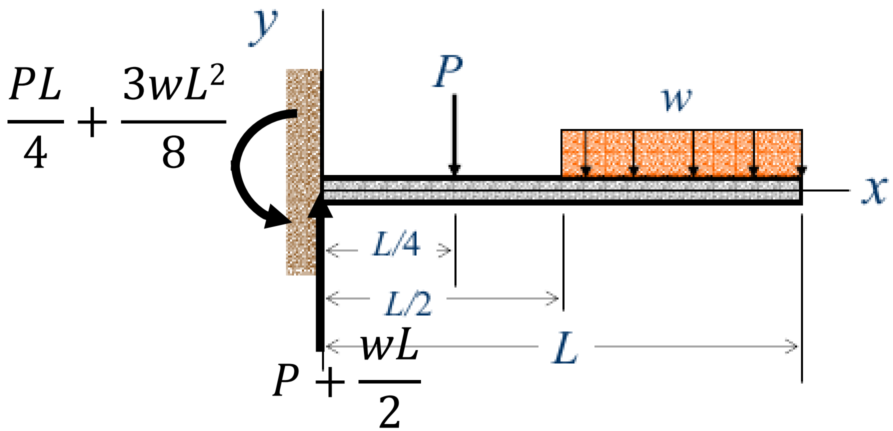
- Deflection = y
- Slope = dy/dx = θ
- Distributed load = w
- Shear force = V
- Bending moment = M

Shear force & bending moment



- *V* and *M* at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions
- The process can be tedious and time-consuming when several intervals and several sets of matching conditions are needed
- The problem is that the shear and moment could only be rarely described by a single analytical function
- Remember the sign conventions for *V* and *M* :

Illustrative example 1

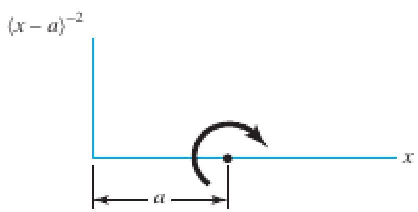
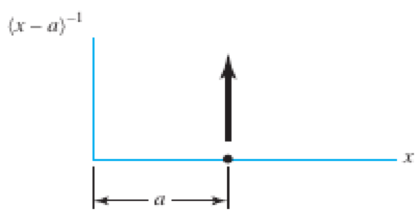
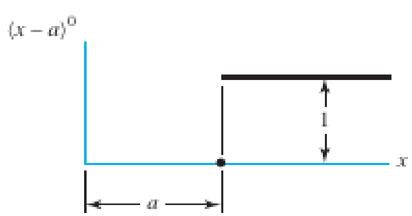
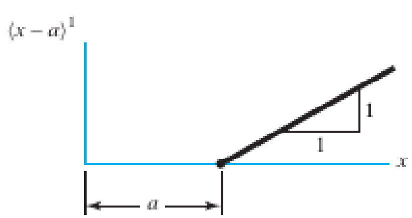


$$V(x)=\begin{cases} P+\dfrac{wL}{2} & \text{for } 0\le x\le L/4 \\ \dfrac{wL}{2} & \text{for } L/4\le x\le L/2 \\ \dfrac{wL}{2}-w\left(x-\dfrac{L}{2}\right) & \text{for } L/2\le x\le L \end{cases}$$

Singularity functions can help reduce the labour by making V or M represented by a single analytical function for the entire length of the beam

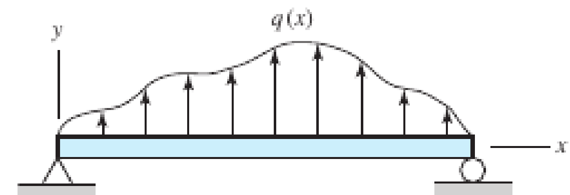
$$M(x)=\begin{cases} -\dfrac{PL}{4}-\dfrac{3wL^2}{8}+Px+\dfrac{wL}{2}x & \text{for } 0\le x\le L/4 \\ -\dfrac{3wL^2}{8}+\dfrac{wL}{2}x & \text{for } L/4\le x\le L/2 \\ -\dfrac{3wL^2}{8}+\dfrac{wL}{2}x-\dfrac{w}{2}\left(x-\dfrac{L}{2}\right)^2 & \text{for } L/2\le x\le L \end{cases}$$

Singularity functions

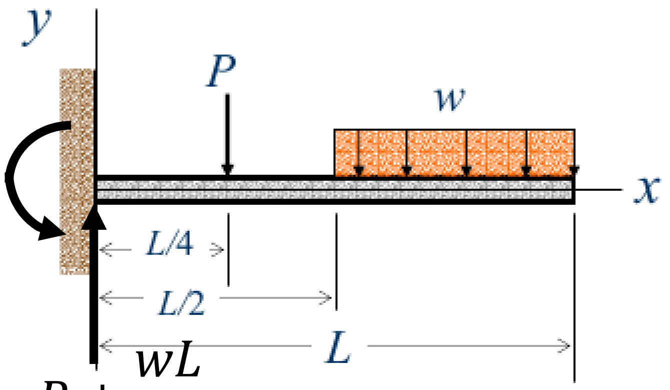
Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x-a\rangle^{-2}=0 \quad x\neq a$ $\langle x-a\rangle^{-2}=\pm\infty \quad x=a$ $\int \langle x-a\rangle^{-2}dx=\langle x-a\rangle^{-1}$
Concentrated force (unit impulse)		$\langle x-a\rangle^{-1}=0 \quad x\neq a$ $\langle x-a\rangle^{-1}=+\infty \quad x=a$ $\int \langle x-a\rangle^{-1}dx=\langle x-a\rangle^{0}$
Unit step		$\langle x-a\rangle^{0}=\begin{cases}0 & x<a\\ 1 & x\geq a\end{cases}$ $\int \langle x-a\rangle^{0}dx=\langle x-a\rangle^{1}$
Ramp		$\langle x-a\rangle^{1}=\begin{cases}0 & x<a\\ x-a & x\geq a\end{cases}$ $\int \langle x-a\rangle^{1}dx=\frac{\langle x-a\rangle^{2}}{2}$

†W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics*, vol. 48, pp. 129–130, 1919.

A singularity function is expressed as $\langle x-a\rangle^n$ where n is any integer (positive or negative) including zero, and a is a constant equal to the value of x at the initial boundary of a specific interval along the beam (note: the singularity functions are for loading q)



Illustrative example 2



$$\frac{PL}{4}+\frac{3wL^2}{8}$$

$$P+\frac{wL}{2}$$

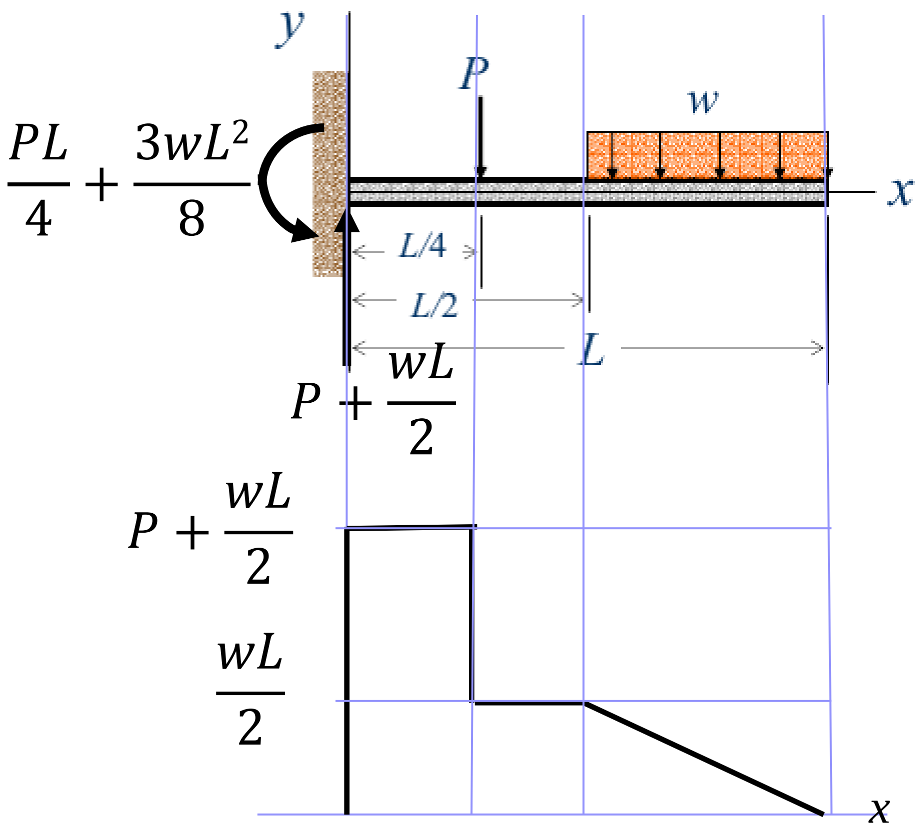
For $0 \le x \le L$

$$q(x) = -\frac{PL}{4}\langle x\rangle^{-2} - \frac{3wL^2}{8}\langle x\rangle^{-2} + P\langle x\rangle^{-1} + \frac{wL}{2}\langle x\rangle^{-1} - P\left\langle x-\frac{L}{4}\right\rangle^{-1} - w\left\langle x-\frac{L}{2}\right\rangle^{0}$$

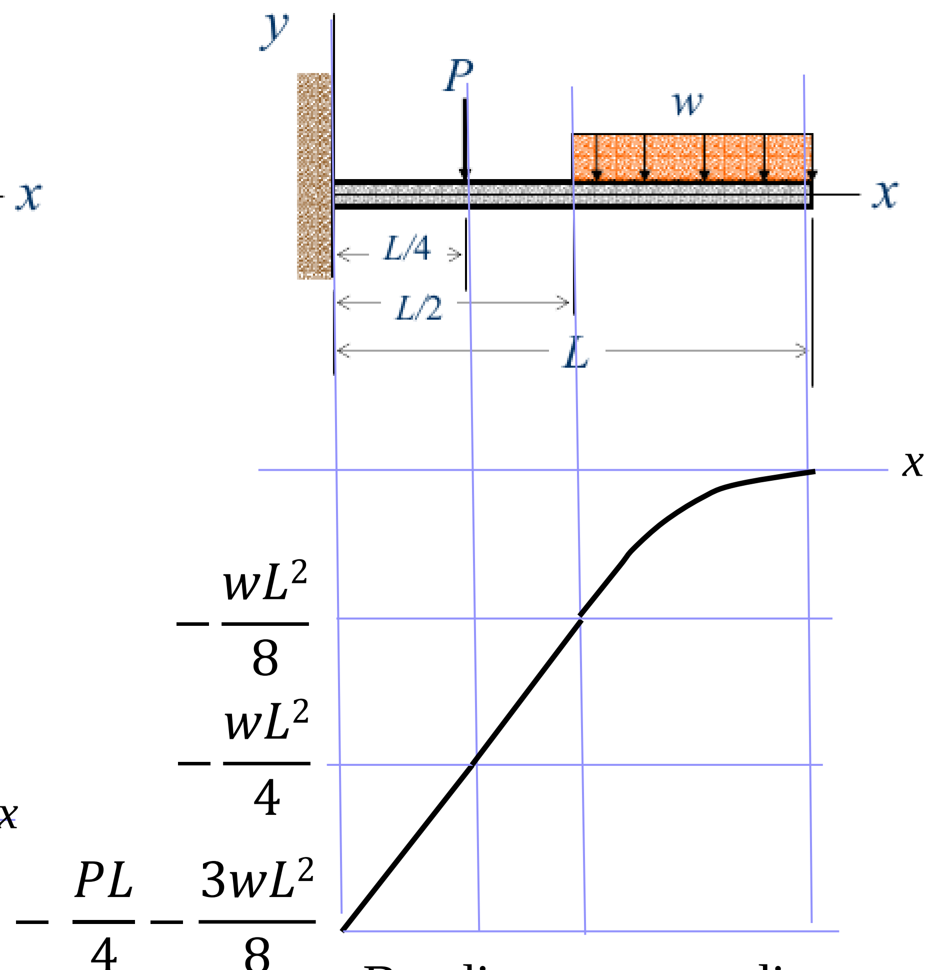
$$V(x) = -\frac{PL}{4}\langle x\rangle^{-1} - \frac{3wL^2}{8}\langle x\rangle^{-1} + P\langle x\rangle^{0} + \frac{wL}{2}\langle x\rangle^{0} - P\left\langle x-\frac{L}{4}\right\rangle^{0} - w\left\langle x-\frac{L}{2}\right\rangle^{1}$$

$$M(x) = -\frac{PL}{4}\langle x\rangle^{0} - \frac{3wL^2}{8}\langle x\rangle^{0} + P\langle x\rangle^{1} + \frac{wL}{2}\langle x\rangle^{1} - P\left\langle x-\frac{L}{4}\right\rangle^{1} - \frac{w}{2}\left\langle x-\frac{L}{2}\right\rangle^{2}$$

Illustrative example 2

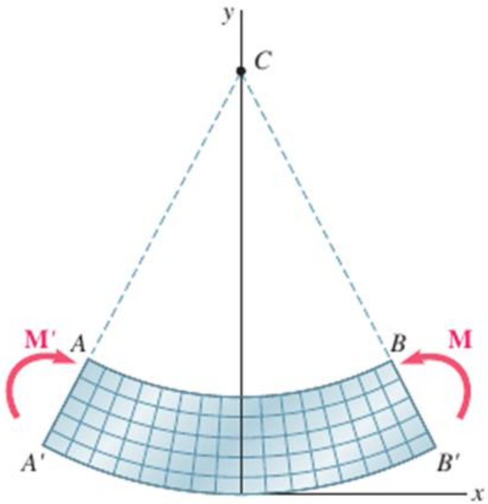


Shear force diagram



Bending moment diagram

Normal stress in beams



- Normal stress due to bending

Normal stress $\sigma = -\frac{My}{I}$ $\qquad |\sigma_{max}| = \frac{Mc}{I} = \frac{M}{S}$

I = second-area moment about z-axis
$S = I/c$ = section modulus

The maximum internal bending moment M in the beam can be found from the bending moment diagram

Max internal bending moment, (Nm)

Max bending stress, Pa

Max distance from NA to outer fiber (m)

Moment of inertia (m^4)

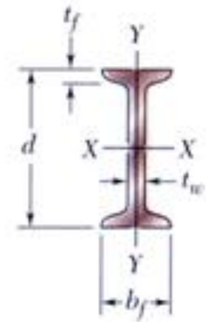
$$|\sigma_{\max}| = \frac{M_{\max} c}{I_z}$$

Or in general: $|\sigma_{\max}| = \frac{M}{S_z}$

Beam section properties

Properties of Rolled-Steel Shapes
(SI Units)

S Shapes
(American Standard Shapes)



Note the change in the x axis definition

Section modulus

Moment of inertia

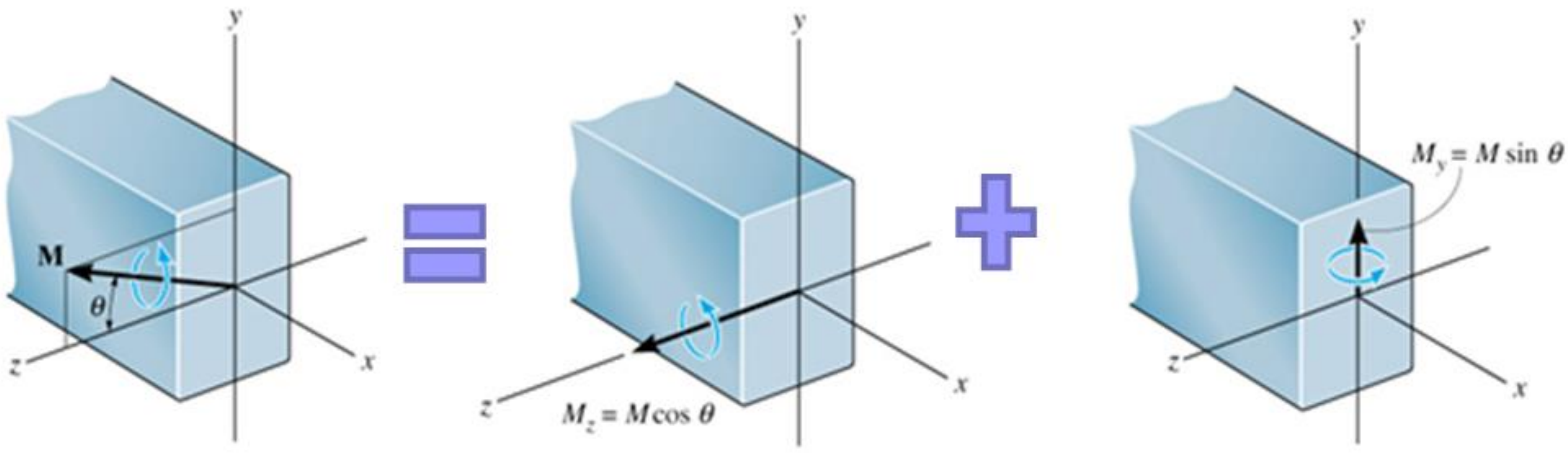
Radius of gyration

Designation†	Area A, mm^2	Depth d, mm	Flange Width b_f, mm	Flange Thickness t_f, mm	Web Thickness t_w, mm	Axis X-X I_x 10^6 mm^4	Axis X-X S_x 10^3 mm^3	Axis X-X r_x mm	Axis Y-Y I_y 10^6 mm^4	Axis Y-Y S_y 10^3 mm^3	Axis Y-Y r_y mm
S610 × 180	22900	622	204	27.7	20.3	1320	4240	240	34.9	341	39.0
158	20100	622	200	27.7	15.7	1230	3950	247	32.5	321	39.9
149	19000	610	184	22.1	18.9	995	3260	229	20.2	215	32.3
134	17100	610	181	22.1	15.9	938	3080	234	19.0	206	33.0
119	15200	610	178	22.1	12.7	878	2880	240	17.9	198	34.0
S510 × 143	18200	516	183	23.4	20.3	700	2710	196	21.3	228	33.9
128	16400	516	179	23.4	16.8	658	2550	200	19.7	216	34.4
112	14200	508	162	20.2	16.1	530	2090	193	12.6	152	29.5
98.3	12500	508	159	20.2	12.8	495	1950	199	11.8	145	30.4
S460 × 104	13300	457	159	17.6	18.1	385	1685	170	10.4	127	27.5
81.4	10400	457	152	17.6	11.7	333	1460	179	8.83	113	28.8
S380 × 74	9500	381	143	15.6	14.0	201	1060	145	6.65	90.8	26.1
64	8150	381	140	15.8	10.4	185	971	151	6.15	85.7	27.1

Two-plane bending

Quite often, in mechanical design, bending occurs in both *xy* and *xz* planes
Considering cross sections with one or two planes of symmetry only, the bending stresses are given by

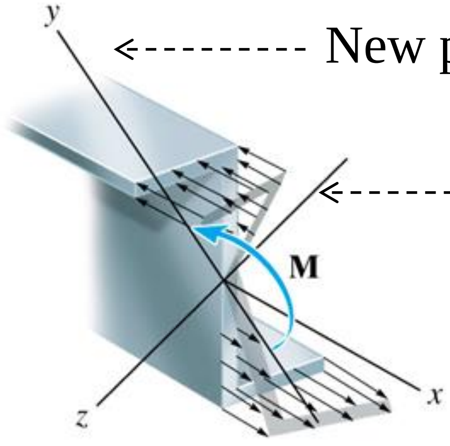
$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$



Asymmetrical cross section

- For non-symmetric cross-sections we must find new principal axes, I_y and I_z Refer to notes (in BB) on how to find the principal axis for non-symmetric cross-sections
- The moments must also be resolved to about the principal axis
- Apply the two-plane analysis about the principal axes

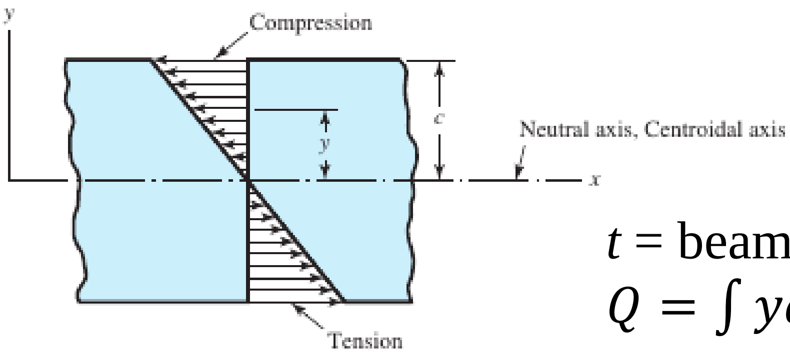
Example of unsymmetrical cross-section:



<---------- New principal y axis

<---------- New principal z axis

Shear stress in beams

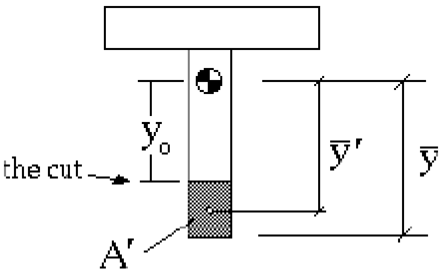


Shear stress $\tau_{ave} = \frac{VQ}{It}$

t = beam thickness

$Q = \int y dA$ = moment of area above t about N.A.

The maximum internal shear force *V* in the beam can be found from the shear force diagram. To find *Q*, Integration may not be needed for simple shapes

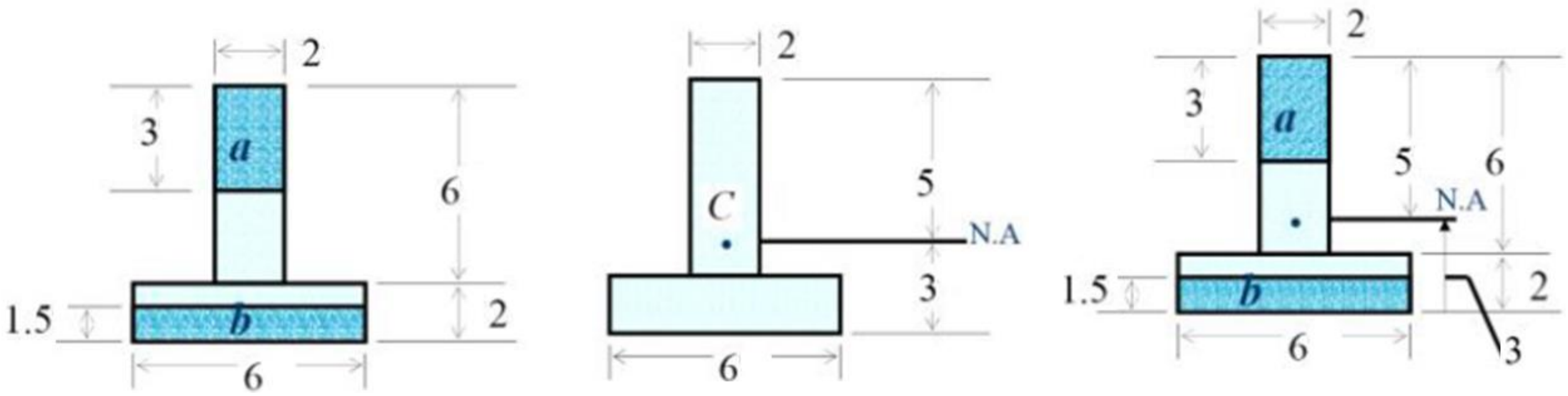


- *Q* is taken about the NA for the entire cross section
- The determination of the shear stress involved a cut area (e.g. bottom area A' below the cut)
- To find Q for simple shapes, we need the cut area about distance between the NA and centroid of cut area (i.e. distance $\overline{y'}$), i.e.

$$Q = \bar{y}' A'$$

Illustrative example 3

Determine the first moment of area *Q* for the areas indicated by the shaded areas "*a*" and "*b*" (all dimensions in mm)



First, locate the neutral axis for the entire area:

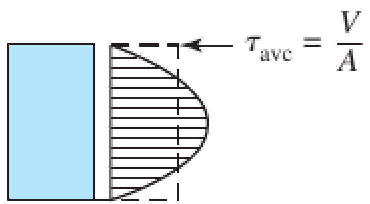
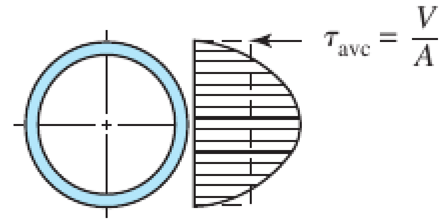
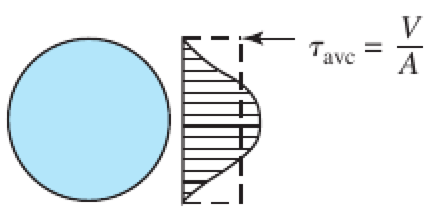
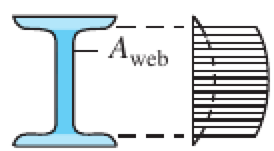
$$y_c = \frac{(1)(2\times6)+(2+3)(2\times6)}{2\times6+2\times6} = 3\text{mm from base}$$

$$Q_a = (5-1.5)[3\times2] = 21\text{mm}^3;$$

$$Q_b = (3-1.5/2)[1.5\times6] = 20.25\text{mm}^3;$$

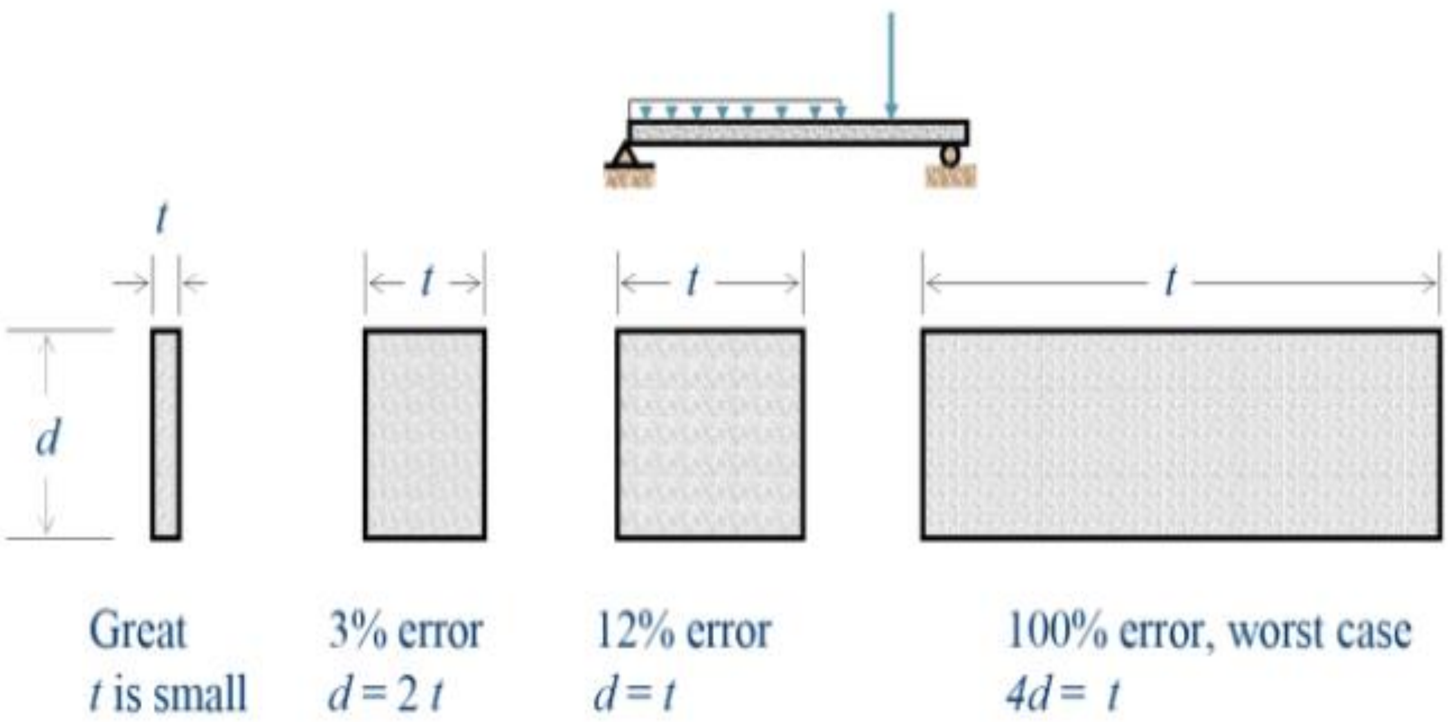
Where will max Q occur?

Shear stress - common sections

Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{max} = \frac{2V}{A}$
 Circular	$\tau_{max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{max} \approx \frac{V}{A_{web}}$

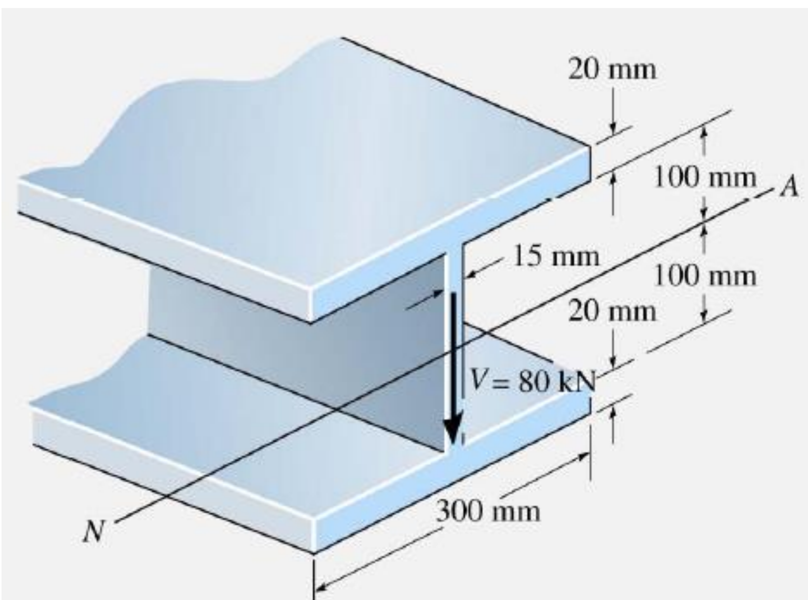
Shear formula accuracy

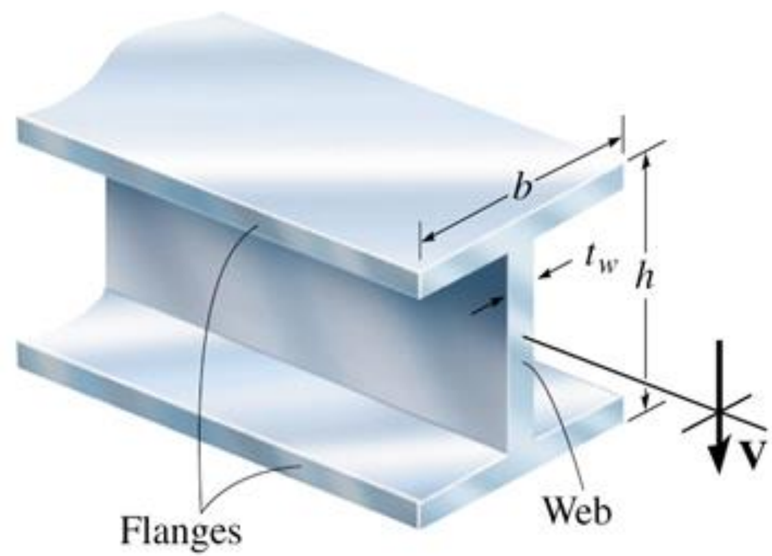
How accurate is the shear stress formula?



Example 1

A steel wide-flange beam has the dimensions shown. If it is subjected to a shear of V = 80kN. Plot the shear-stress distribution over the beam's cross-sectional area

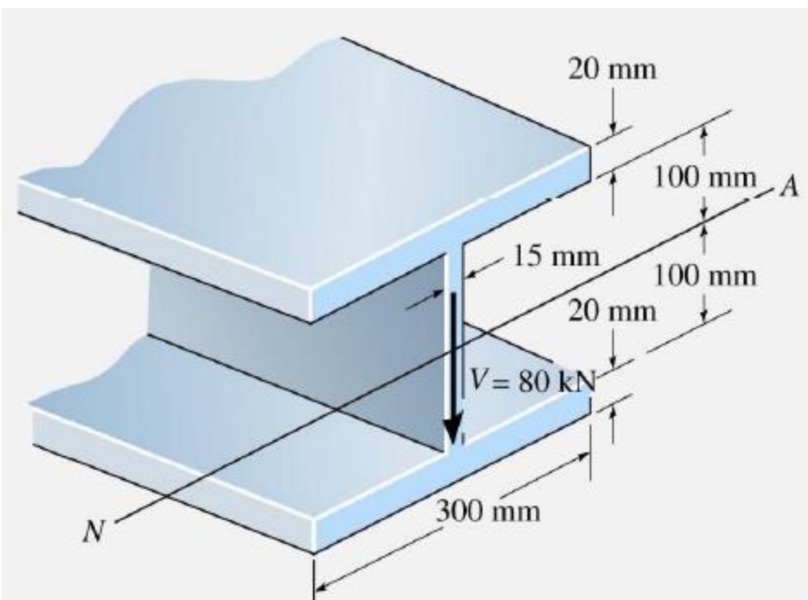


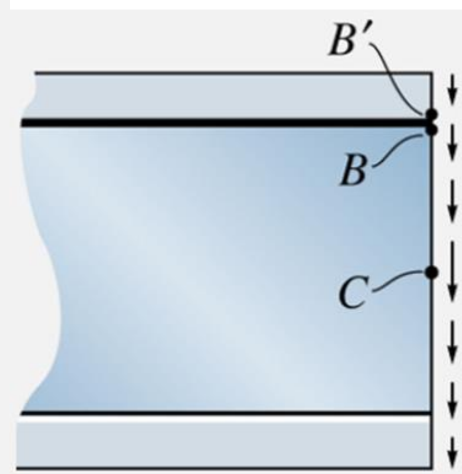


Example 1

$$I = \left[\frac{1}{12}(0.015)(0.2)^3\right] + 2\left[\frac{1}{12}(0.3)(0.02)^3 + 0.3(0.02)(0.11)^2\right]$$
$$= 155.6(10^{-6})\text{m}^4;$$

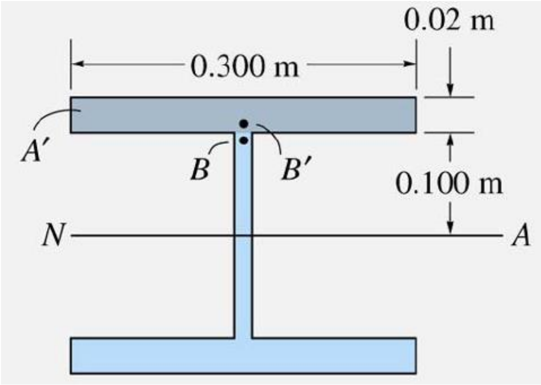
Due to symmetry, only shear stresses at points B', B, and C have to be found





B' belongs to the flange; B belongs to the web (B' and B are considered to be at the flange-web junction with $Q_{B'}=Q_B$ but $t_{B'}\neq t_B$)

Example 1



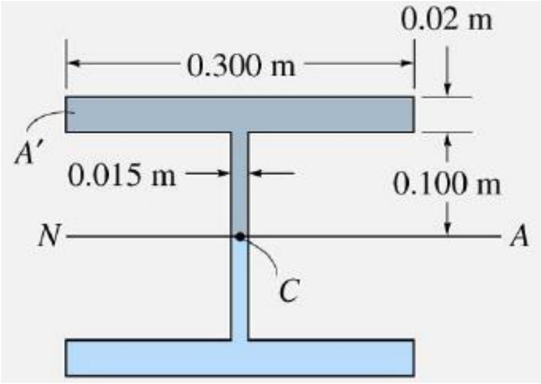
$Q_B = Q_{B'} = \bar{y}'A' = 0.11(0.3)(0.02) = 0.66(10^{-3})\text{m}^3$;

For point B’, $t_{B'}$=0.3m and

$$\tau_{B'} = \frac{VQ_{B'}}{It_{B'}} = \frac{(80000)[0.66(10^{-3})]}{155.6(10^{-6})(0.3)} = 1.13\text{MPa}$$

For point B, t_B=0.015m and

$$\tau_B = \frac{VQ_B}{It_B} = \frac{(80000)[0.66(10^{-3})]}{155.6(10^{-6})(0.015)} = 22.6\text{MPa}$$



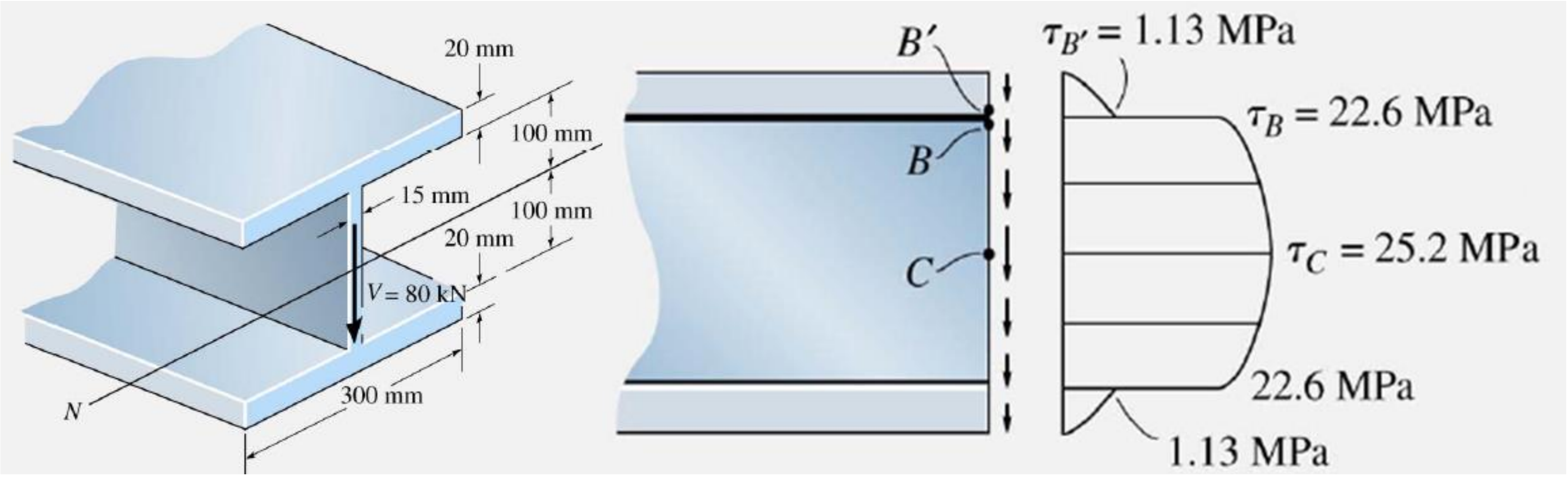
For point C, t_C=0.015m and

$$Q_C = \sum \bar{y}'A' = 0.11(0.3)(0.02) + [0.05](0.015)(0.1)$$

$$Q_C = 0.735(10^{-3})\text{m}^3;$$

$$\tau_C = \frac{VQ_C}{It_C} = \frac{(80000)[0.735(10^{-3})]}{155.6(10^{-6})(0.015)} = 25.2\text{MPa}$$

Example 1



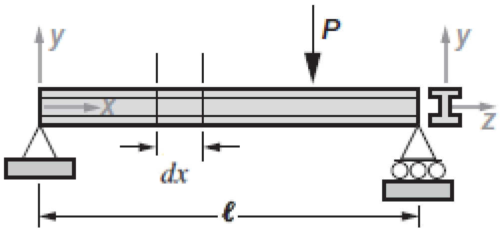
Where will max τ occur?

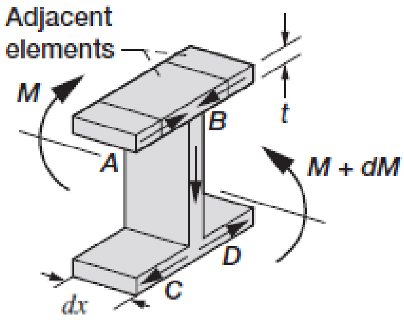
Where will sudden changes in τ occur?

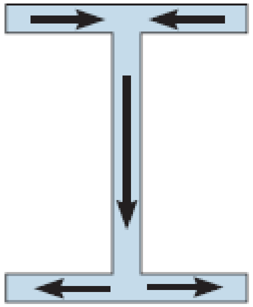
Shear flow

$$q = \tau t = \frac{VQ}{I}$$

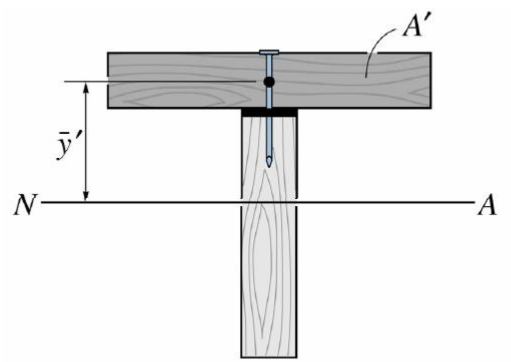
- Shear flow q (N/m) is a measure of the shear force per unit length along the longitudinal axis of the beam
- It is used to determine the shear force developed in fasteners and glue that hold various segments of the beam together
- Value of q changes over the cross section, since Q will be different for each area segment
- Shear flow will always act parallel to the walls of the member, since section on which q is calculated is taken perpendicular to the wall



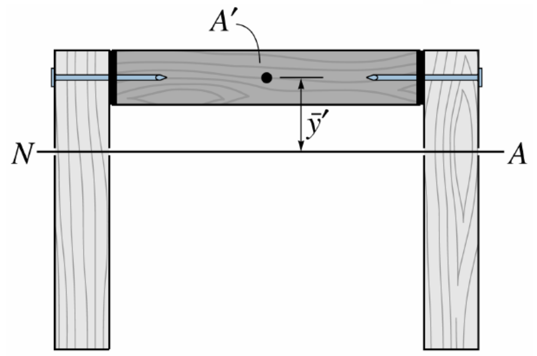




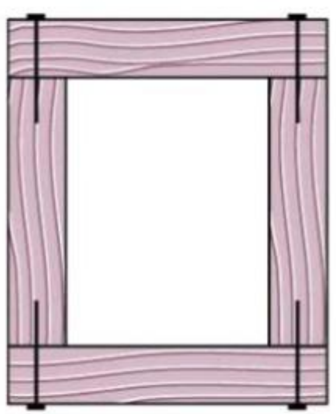
Shear flow in built-up members



Shear flow q will be resisted by a single fastener

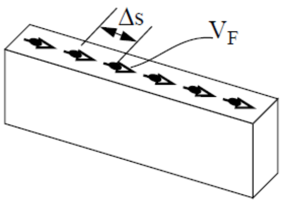


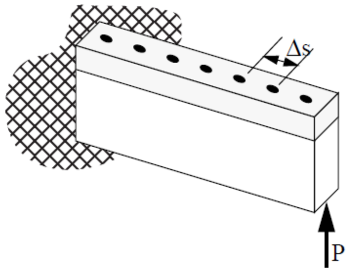
Shear flow q will be resisted by two fasteners



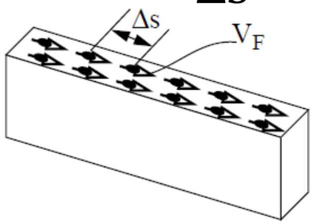
To design the fasteners, we need to know the shear force to be resisted by the fastener along the member's length

$$q = \frac{V_F}{\Delta s}$$



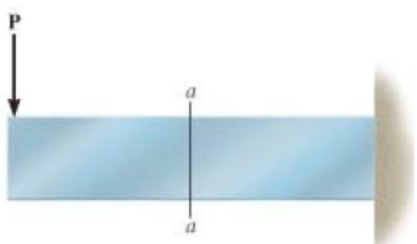


$$q = \frac{2V_F}{\Delta s}$$

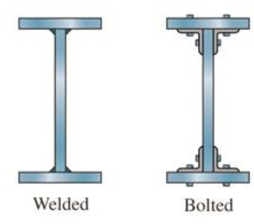


Prismatic beam design

- Beams are structural members designed to support loads applied perpendicular to the longitudinal axis
- With beams there are two stresses to deal with normal stress, σ, and shear stress, τ
- Bending stress σ is generally the critical stress (i.e. for long beams where L/h >10). Hence, beam design usually involves finding the V_{max} and M_{max} from the shear force and bending moment diagrams and then using M_{max} to determine the material and design the beam cross section based on σ_{allow}.



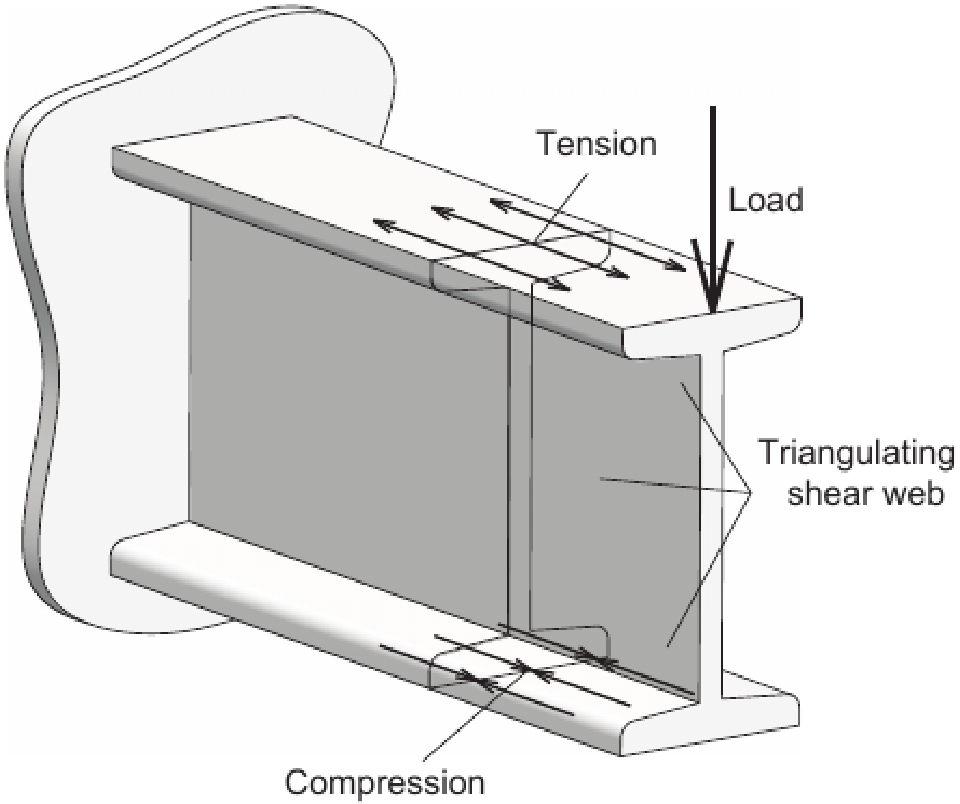
- The V_{max} is then used to ensure that the $\tau < \tau_{allow}$.
- If there are fasteners, then fastener strength (or size) will need to be determined using shear flow q



Steel plate girders


Wooden box beam

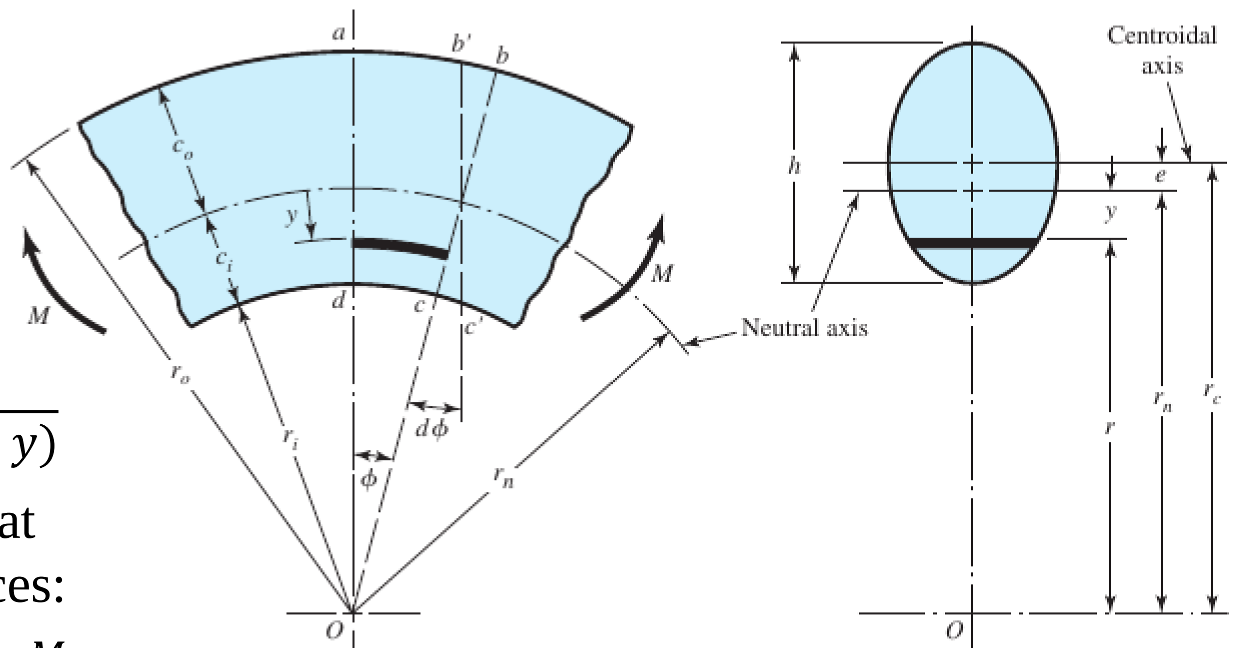
Prismatic beam design



Curved Beams

- For curved beams, we assumes that the cross-sectional area is constant
- The cross section has an axis of symmetry in the plane of bending
- The location of the neutral axis with respect to the center of curvature O is given by r_n:

$$r_n = \frac{A}{\int \frac{dA}{r}}$$



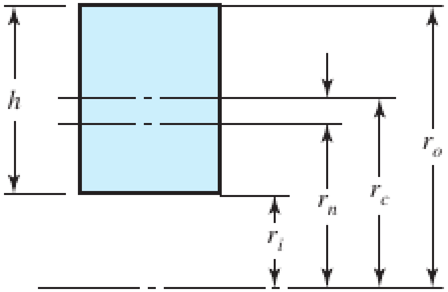
Stress distribution is given by

$$\sigma = \frac{My}{Ae(r_n - y)}$$

Critical stress occurs at inner and outer surfaces:

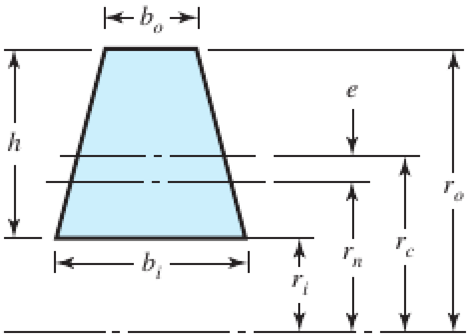
$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \sigma_o = \frac{Mc_o}{Aer_o}$$

Curved Beams



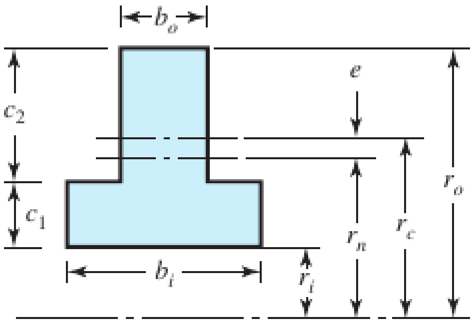
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{h}{3}\frac{b_i + 2b_o}{b_i + b_o}$$

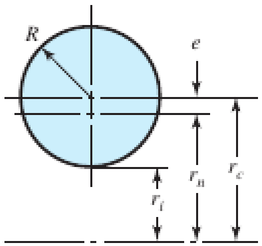
$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

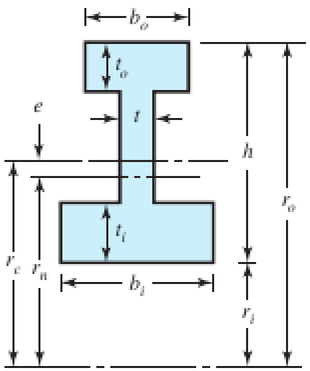
$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$

Curved Beams



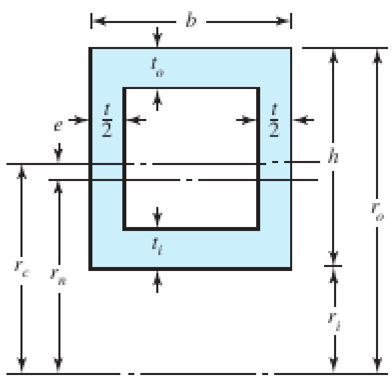
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2\left(r_c - \sqrt{r_c^2 - R^2}\right)}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

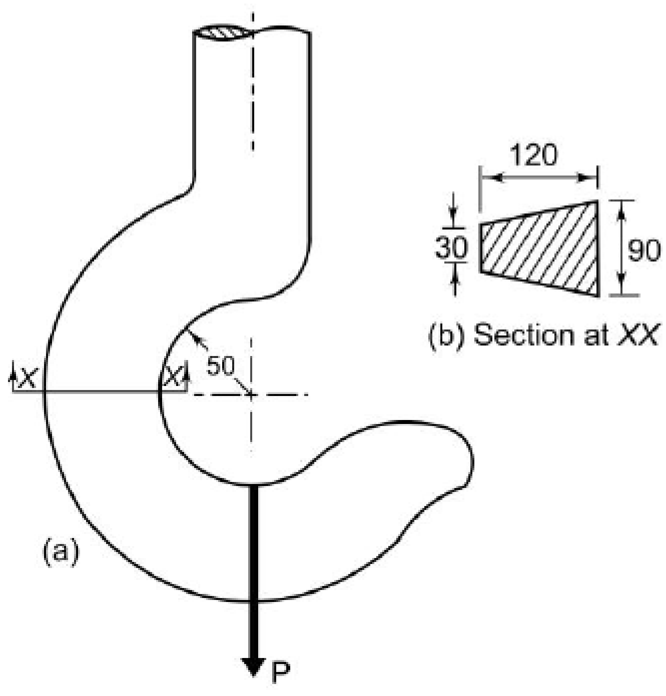
$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln\frac{r_i + t}{r_i} + t \ln\frac{r_o - t_o}{r_i + t_i} + b_o \ln\frac{r_o}{r_o - t_o}}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b\left(\ln\frac{r_i + t_i}{r_i} + \ln\frac{r_o}{r_o - t_o}\right) + t \ln\frac{r_o - t_o}{r_i + t_i}}$$

Example 2



A crane hook having an approximate trapezoidal cross-section is shown (all dimensions are in mm). It is made of plain carbon 45C8 steel (S_{yt} = 380MPa) and the factor of safety is 3.5; Determine the load carrying capacity of the hook

The permissible tensile strength is

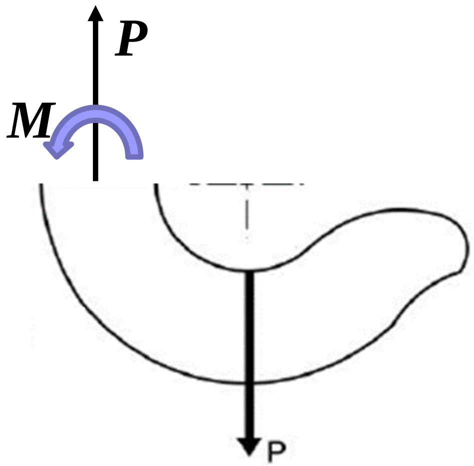
$$\sigma_{max} = \frac{S_{yt}}{(fs)} = 108.57\text{MPa};$$

Use known cross section to determine r_c and r_n given r_i=50mm; b_i=90mm; b_o=30mm; h=120mm

$$r_c = r_i + \frac{h}{3}\frac{b_i+2b_o}{b_i+b_o} = 100\text{mm; and } r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o/r_i)} = 89.18\text{mm}$$

Note: c_i=(89.18 - 50) = 39.18mm

Example 2



Given r_i=50mm; b_i=90mm; b_o=30mm; h=120mm;
Found $\sigma_{max} = 108.57$MPa;
$r_c = 100$mm and $r_n = 89.18$mm; c_i=39.18mm
Eccentricity $e = r_c - r_n$ =10.82mm
Area $A = \frac{1}{2}(b_o + b_i)h$ =7200mm^2;

Direct tensile stress due to P is
$\sigma_t = \frac{P}{A} = 1.39P(10^{-4})$MPa;
Critical tensile stress due to bending occur at inner fibre: $\sigma_{bi} = \frac{Mc_i}{Aer_i} = \frac{Pr_c c_i}{Aer_i} = 10.06P(10^{-4})$MPa;

Using the principle of superposition: $\sigma_t + \sigma_{bi} = \sigma_{max}$ or
$11.45P(10^{-4}) = 108.57$; Hence P = 94.8kN