# Design of Car Suspension System

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# Acknowledgments

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# 1 Introduction

With increased requirements for vehicle performances, vehicle suspension systems are of importance for contributing to the cars handling and keeping vehicle occupants comfortable and reasonably well isolated from road noise, bumps, vibrations, etc (Bastow et al., 2004). The stability of vehicle when it moving above different terrain is very important criterion in addition to ride comfort of passengers. Traffic accidents in the world wide lead to losing people their lives or suffer a non–fatal injury. The ride comfort and road handling of automobiles are majorly achieved by their suspension systems, which represent the connection between automobile body and road (Mitra et al., 2016). The comfortable ride we enjoy today is largely attributed to modern advances in car suspension systems. Designing an effective suspension system is as much an art as it is a science. Lots of compromises must be made between ride quality and handling performance, the former being how comfortable the car is and the latter, how well it remains stable and controllable at speed (Nagarkar et al., 2018). In reality, the suspension system has a massive amount to do and the components have to withstand an enormous amount of stress compared to other major systems in a car. The suspension system as shown in Figure 1 is located between the frame and the wheels and serves multiple important purposes. Ideally, well-tuned suspension will absorb bumps and other imperfections in the road so the people inside the car can travel comfortably. While this is very important from a passenger's perspective, the driver will notice certain other attributes of the suspension system. This system also is responsible for keeping the wheels on the ground as much as possible (Shelke et al., 2018).

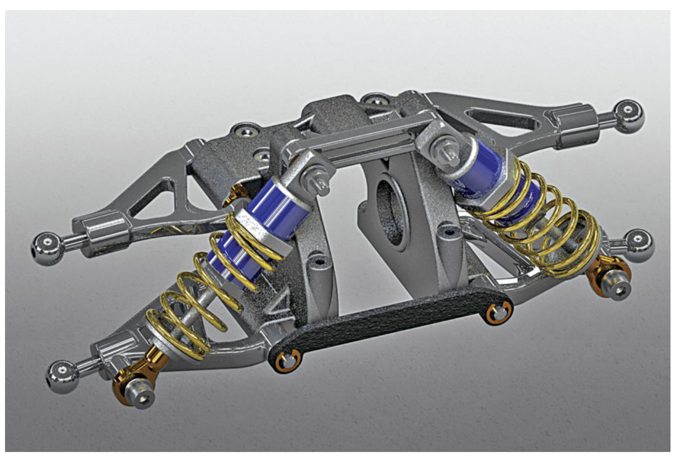

**Figure 1.** Suspension System in a Real Car.

Two principal components or parts–the springs and dampening mechanisms–are the basic construction of a modern car suspension, Of course, there may be other parts such as bushings, suspension strut, and others (Karnopp & Margolis, 1984). The spring gives the car the ability to compensate for any irregularity on the surface of the road. It also serves to support any additional weight on the vehicle without excessive sagging. The spring is also that part of the suspension that keeps it at a predetermined height. While a spring can help absorb the energy off bumps on the road, without a dampening mechanism to help control or dissipate this up and down energy, the people inside the car will be left with a vehicle that will continue to oscillate with each bump until such time that the energy has fully dissipated (Majjad, 1997). Therefore, the damper plays a important role in dissipating the oscillating energy that caused by irregularity on the

surface of road.
In this project, I will study the motion response of a given car suspension system, conduct the analysis of car suspension systems using different analytical, numerical, and Simulink models in Section 2, and explore the means of vibration control for the suspension system, including passive system, active system, and semi–active system in Section 3.

# 2 Analysis of 2 DoF Car Suspension Systems

In this section, I will research the response of the car suspension system due to the input the sine function from the road. First, I will derive the differential equations of motion and obtain the mass, stiffness, and damping matrices, and the force vector. Second, according to the result in the first step, I will determine the displacement transmissibility, force transmissibility, natural frequencies and mode shapes, etc. for the systems using MATLAB. In depth, I will use MATLAB to compute the responses (i.e. $x_1$ versus time and $x_2$ versus time) of the system with and without the damper. Afterwards, I will vary the values for $c$ and $k_1$ to examine the effects and the quality of the response, including displacement transmissibility, and force transmissibility, to the input road surface, from which I will determine the values for $c$ and $k_1$ that would produce the "best" suspension system.

## 2.1 Derivation of Differential Equations of Motion

For the model Professor Fok provided, which is shown in Figure 2, I choose the displacements of the two masses $x_1$ and $x_2$ as the generalized coordinates. The static equilibrium positions of $m_1$ and $m_2$ are set as the coordinate origins. Assume

$$x_1 > x_2 > z > 0 \tag{1}$$

which implies that the springs are in tension and

$$\dot{x}_1 > \dot{x}_2 > \dot{z} > 0 \tag{2}$$

The free-body diagrams of $m_1$ and $m_2$ are shown in the right side of Figure 2. According to the assumption, the mass $m_1$ moves faster than the mass $m_2$, and the elongation of the spring $k_1$ is $x_1 - x_2$. The force exerted by the spring $k_1$ on the mass $m_1$ is downward as it tends to restore to the undeformed position. Because of Newton's third law, the force exerted by the spring $k_1$ on the mass $m_2$ has the same magnitude, but opposite in direction. Other spring forces and damping forces can be determined using the same logic. Note that the gravitational forces, $m_1 g$ and $m_2 g$, are not included in the free-body diagrams.
Applying Newton's second law to the masses $m_1$ and $m_2$, respectively, gives

$$+\uparrow x : \sum F_x = m a_x \tag{3}$$

$$-k_1(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1 \tag{4}$$

$$k_1(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) - k_2(x_2 - y) = m_2 \ddot{x}_2 \tag{5}$$

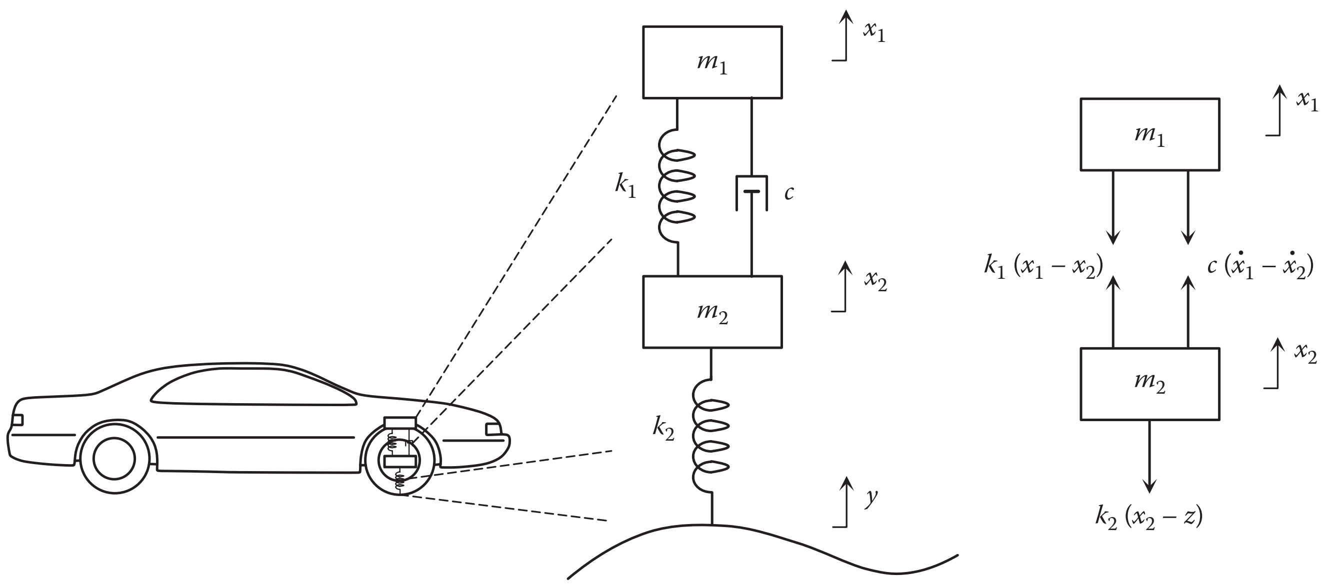


**Figure 2.** Simplified Suspension System of Car Model.

Rearranging the equations into the standard input–output form,

$$\begin{cases} m_1\ddot{x}_1 + c\dot{x}_1 - c\dot{x}_2 + k_1x_1 - k_1x_2 = 0 \\ m_2\ddot{x}_2 - c\dot{x}_1 + c\dot{x}_2 - k_1x_1 + (k_1+k_2)x_2 = k_2y \end{cases} \tag{6}$$

which can be expressed in second-order matrix form as shown in Equation 7.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & k_2 \end{bmatrix}\begin{Bmatrix} 0 \\ y \end{Bmatrix} \tag{7}$$

Because my student ID number is 2018141521058, which is ending with 8, the parameter I get for the car model is that the vehicle is travelling with a maximum velocity $v = 100$ km/h on a sinusoidal road surface with amplitude $Y = 0.019$ m, and a wavelength of $\lambda = 5.5$ m. Assume $m_1 = 1010$ kg, $m_2 = 76$ kg, $k_1 = 31110$ N/m, $k2 = 321100$ N/m, and $c = 4850$ N · s/m.
There are many reasons caused the disturbance such as road surface terrain, aerodynamic forces, non-uniformity of the wheel, tire assembly, and even or barking forces (Tsubokura et al., 2010). According to the relationship among velocity, wavelength, and frequency, I can derive the frequency of the road surface irregularity as shown in Equation 8.

$$f = \frac{v}{\lambda} = \frac{\frac{100}{3.6}\ \mathrm{m/s}}{5.5\ \mathrm{m}} = 5.0505\ \mathrm{Hz} \tag{8}$$

Then, we can know that the input frequency $\omega$ is equal to

$$\omega = 2\pi f = 2\pi \times 5.0505\ \mathrm{Hz} = 31.7333\ \mathrm{rad/s} \tag{9}$$

We also know the amplitude of input signal, according to which we can derive the entire input function $y(t)$ as shown in Equation.

$$y(t) = Y \sin \omega t = 0.019 \sin 31.7333t \tag{10}$$

Substituting these parameters into Equation 6 and 7 yields that the differential equations of motion and the matrix form of motion are shown in Equation 11 and 12, respectively.

$$\begin{cases} 1010\ddot{x}_1 + 4850\dot{x}_1 - 4850\dot{x}_2 + 31110x_1 - 31110x_2 = 0 \\ 76\ddot{x}_2 - 4850\dot{x}_1 + 4850\dot{x}_2 - 31110x_1 + (31110 + 321100)x_2 = 321100 \times 0.019 \sin 31.7333t \end{cases} \tag{11}$$

$$\begin{aligned} &\begin{bmatrix} 1010 & 0 \\ 0 & 76 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4850 & -4850 \\ -4850 & 4850 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 31110 & -31110 \\ -31110 & 31110 + 321100 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -1 & 321100 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.019 \sin 31.7333t \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6101 \sin 31.7333t \end{Bmatrix} \end{aligned} \tag{12}$$

From Equation 12, I can obtain the mass, stiffness, and damping matrices, and the force vector as shown in Equation 13, 14, 15, and 16, respectively.

$$M = \begin{bmatrix} 1010 & 0 \\ 0 & 76 \end{bmatrix} \tag{13}$$

$$K = \begin{bmatrix} 31110 & -31110 \\ -31110 & 352210 \end{bmatrix} \tag{14}$$

$$C = \begin{bmatrix} 4850 & -4850 \\ -4850 & 4850 \end{bmatrix} \tag{15}$$

$$F = \begin{Bmatrix} 0 \\ 6101 \sin 31.7333t \end{Bmatrix} \tag{16}$$

Note that the gravity terms in this model do not appear in the equations of motion, because the static equilibrium positions are chosen as the coordinate origins.

In depth, I also derive the state-space representation of the system. Note that the input to the system is the road surface irregularity $y$. The state, the input, and the output vectors are

$$x = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}, \ u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ y(t) \end{Bmatrix}, \ y = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{17}$$

The state-variable equations are then obtained as

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 = \ddot{x}_1 &= -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 - \frac{c}{m_1}\dot{x}_1 + \frac{c}{m_1}\dot{x}_2 \\ &= -\frac{k_1}{m_1}x_1 + \frac{k_1}{m_1}x_2 - \frac{c}{m_1}x_3 + \frac{c}{m_1}x_4 \\ \dot{x}_4 = \ddot{x}_2 &= \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 + \frac{c}{m_2}\dot{x}_1 - \frac{c}{m_2}\dot{x}_2 + \frac{k_2}{m_2}y(t) \\ &= \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 + \frac{c}{m_2}x_3 - \frac{c}{m_2}x_4 + \frac{k_2}{m_2}u_2 \end{aligned} \tag{18}$$

The output equation is

$$y = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \tag{19}$$

Thus, the state-space representation is

$$\begin{aligned} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{c}{m_1} & \frac{c}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{c}{m_2} & -\frac{c}{m_2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{k_2}{m_2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \end{aligned} \tag{20}$$

Then, I use MATLAB to get the transfer function of input irregularity of the road and the output $x_1$ and $x_2$, which is shown in Appendices B, from which we can get the results as shown in Equation 21 and 22.

$$\frac{X_1}{Y} = \frac{2.029 \times 10^4 s + 1.301 \times 10^5}{s^4 + 68.62s^3 + 4665s^2 + 2.029 \times 10^4 s + 1.301 \times 10^5} \tag{21}$$

$$\frac{X_1}{Y} = \frac{4225s^2 + 2.029 \times 10^4 s + 1.301 \times 10^5}{s^4 + 68.62s^3 + 4665s^2 + 2.029 \times 10^4 s + 1.301 \times 10^5} \tag{22}$$

## 2.2 Determine the Properties of Suspension System

Using mass, stiffness, and damping matrices, and the force vector, I can calculate the natural frequencies and mode shapes using the MATLAB code as shown in Appendices A. From the result I get from the code, I can obtain the natural frequencies of the $x_1$ and $x_2$ with the damper as shown in Equation 23.

$$\omega_1 = 5.2976 \text{ rad/sec and } \omega_2 = 68.0961 \text{ rad/sec} \tag{23}$$

The plot for mode shapes for the suspension system is shown in Figure 3. From the mode shapes diagram, I can observe that for mode 1, $x_1$ will vibrate crazily compared to $x_2$, which means the people in the car will feel uncomfortable if there is only mode 1. By contrast, for mode 2, $x_1$ hardly vibrates compared to $x_2$. However, we cannot know total shape of the system response just using mode shape. We need to combine the initial condition to derive the response equations for $x_1$ and $x_2$ to obtain the shape of response.

Next, I will conduct the derivation of displacement transmissibility and force transmissibility for general 2 DoF damped system.

We can use complex algebra to represent the harmonic external forces as $F_i(t) = F_{i0}e^{j\omega t}$ for $i = 1, 2$ and $\omega$ is the forcing frequency.

For the 2-DoF damped system:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \cdot e^{j\omega t} \tag{24}$$

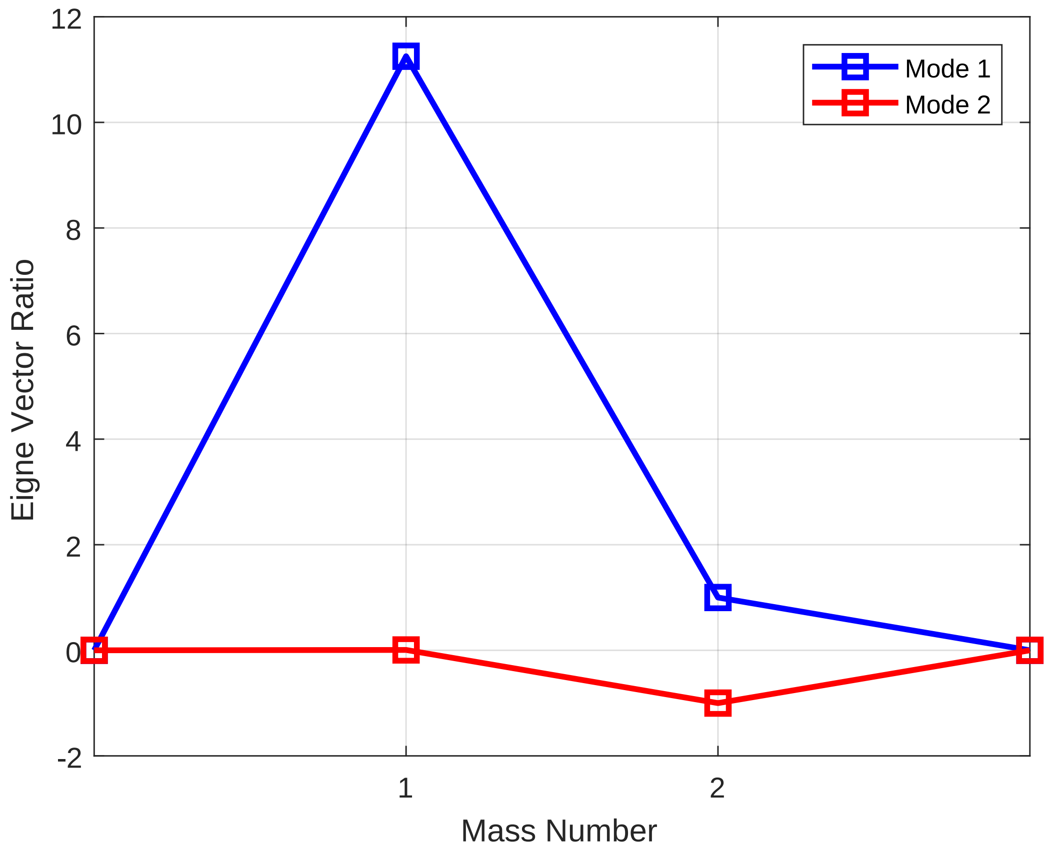


**Figure 3.** Mode shapes for the Suspension System.

The steady state response will have the form $x_i(t) = X_i e^{j\omega t}$.
Note that $\dot{x}_i(t) = j\omega X_i e^{j\omega t}$ and $\ddot{x}_i(t) = -\omega^2 X_i e^{j\omega t}$.
Substituting these back into the equations of motion:

$$-\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} + j\omega \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} e^{j\omega t} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \cdot e^{j\omega t} \tag{25}$$

Simplifying Equation 25 above yields that

$$\begin{bmatrix} -\omega^2 m_{11} + j\omega c_{11} + k_{11} & -\omega^2 m_{12} + j\omega c_{12} + k_{12} \\ -\omega^2 m_{21} + j\omega c_{21} + k_{21} & -\omega^2 m_{22} + j\omega c_{22} + k_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \tag{26}$$

The equation can be rewritten as:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \Longleftrightarrow [Z(j\omega)] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \tag{27}$$

where the mechanical impedance is defined as (for $r, s = 1, 2$)

$$Z_{rs}(j\omega) = -\omega^2 m_{rs} + j\omega c_{rs} + k_{rs} \tag{28}$$

We can solve for displacement as shown below:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [Z(j\omega)]^{-1} \begin{bmatrix} F_{10} \\ F_{20} \end{bmatrix} \tag{29}$$

In this car suspension model, the displacement transmissibility and the force transmissibility are defined as Equation 30 and 31.

$$\begin{aligned} \text{displacement transmissibility for } m_1 &= \frac{X_1}{Y} = k_2 \cdot \frac{X_1}{F_{20}} = k_2 \cdot \left| \frac{-k_{12} + \omega^2 m_{12} - j\omega c_{12}}{\det\left([Z(j\omega)]\right)} \right| \\ \text{displacement transmissibility for } m_2 &= \frac{X_2}{Y} = k_2 \cdot \frac{X_2}{F_{20}} = k_2 \cdot \left| \frac{k_{11} - \omega^2 m_{11} + j\omega c_{11}}{\det\left([Z(j\omega)]\right)} \right| \end{aligned} \tag{30}$$

$$\begin{aligned} \text{force transmissibility for } m_1 &= \frac{k_1(X_1 - X_2)}{F_{20}} \\ &= k_1 \cdot \left\{ \left| \frac{-k_{12} + \omega^2 m_{12} - j\omega c_{12}}{\det\left([Z(j\omega)]\right)} \right| - \left| \frac{k_{11} - \omega^2 m_{11} + j\omega c_{11}}{\det\left([Z(j\omega)]\right)} \right| \right\} \\ \text{force transmissibility for } m_2 &= \frac{k_2(X_2 - Y) - k_1(X_1 - X_2)}{F_{20}} \\ &= k_2 \cdot \left\{ \left| \frac{k_{11} - \omega^2 m_{11} + j\omega c_{11}}{\det\left([Z(j\omega)]\right)} \right| - \frac{1}{k_2} \right\} - \\ &\quad k_1 \cdot \left\{ \left| \frac{-k_{12} + \omega^2 m_{12} - j\omega c_{12}}{\det\left([Z(j\omega)]\right)} \right| - \left| \frac{k_{11} - \omega^2 m_{11} + j\omega c_{11}}{\det\left([Z(j\omega)]\right)} \right| \right\} \end{aligned} \tag{31}$$

Using the displacement transmissibility and the force transmissibility I derived, I write the MATLAB code as shown in Appendices C. After runing the code, I can find out the result: the displacement transmissibility for mass 1 and 2 are equal to $0.169$ and $1.077$, respectively, and the force transmissibility for mass 1 and 2 are equal to $0.088$ and $0.165$, respectively.

## 2.3 Response of Suspension System

Using the code shown in Appendices D.1 and D.3, I can plot the numeric response of $x_1$ and $x_2$ with damper as shown in Figure 4.

In repect to the displacement transmissibility and the force transmissibility, from Figure 4, I can obtain the maximum displacement for steady-state response $X_1$ and $X_2$. Then, I can use Equation 32 and 33 to find the displacement transmissibility and the force transmissibility.

$$\begin{aligned} \text{displacement transmissibility for } m_1 &= \frac{X_1}{Y} = \frac{0.003406}{0.019} = 0.1793 \\ \text{displacement transmissibility for } m_2 &= \frac{X_2}{Y} = \frac{0.02276}{0.019} = 1.1979 \end{aligned} \tag{32}$$

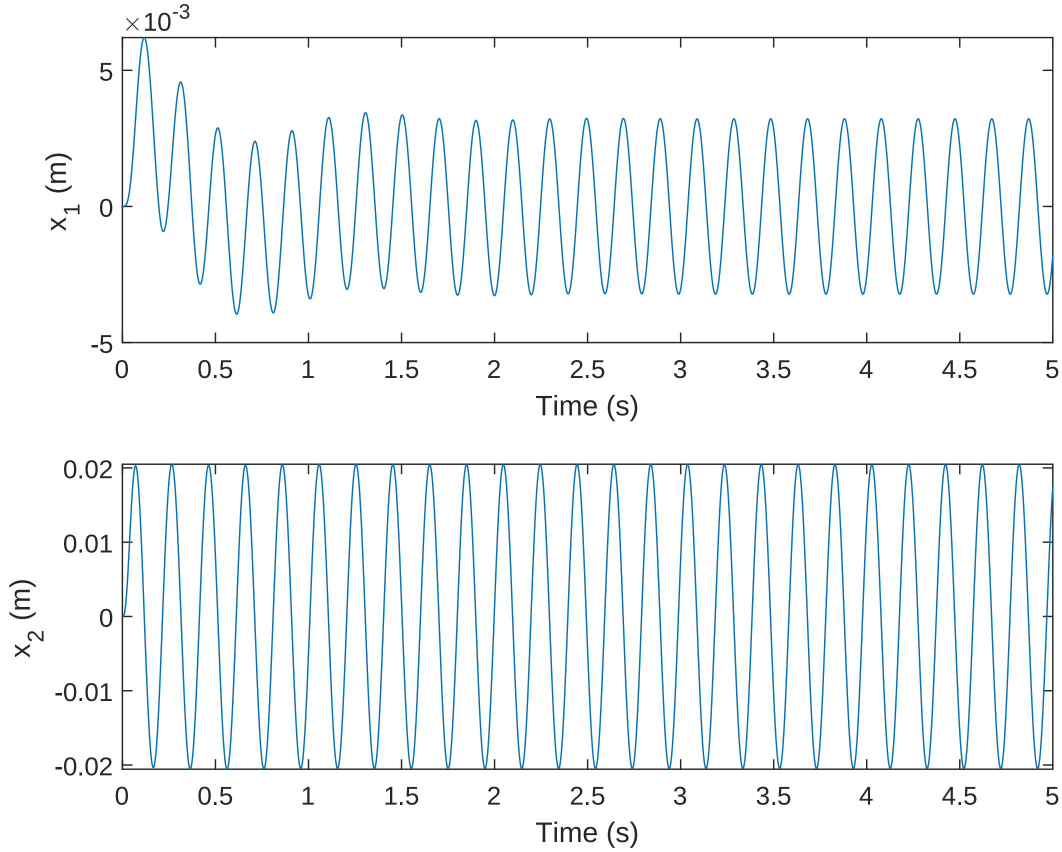


**Figure 4.** Numeric Response of Car Model with Damper.

$$\begin{aligned}
\text{force transmissibility for } m_1 &= \frac{k_1(X_1 - X_2)}{k_2 Y} = \left|\frac{31110 \times (0.003406 - 0.02276)}{321100 \times 0.019}\right| = 0.09869 \\
\text{force transmissibility for } m_2 &= \frac{k_2(X_2 - Y) - k_1(X_1 - X_2)}{k_2 Y} \\
&= \left|\frac{321100 \times (0.02276 - 0.019) - 31110 \times (0.003406 - 0.02276))}{321100 \times 0.019}\right| \\
&= 0.2965
\end{aligned} \tag{33}$$

Comparing the result I get from the numeric solution with that I derive from my calculation, I can know that there is a little bit difference between them. But they are already very close to each other. Also, we notice that the displacement transmissibility and the force transmissibility of mass 1 are relatively low, which are almost near $10\%$. This means the displacement and the force transmitted due to the irregularity of the road to the driver inside the car are small compared to that of the road, which ensures the comfort of the driver in the car.

Then, I use Simulink to confirm my numeric solution. The Simulink block diagram is shown in Appendices E.1. The results of Simulink response for $x_1$ and $x_2$ are shown in Figure 5 and 6, respectively.

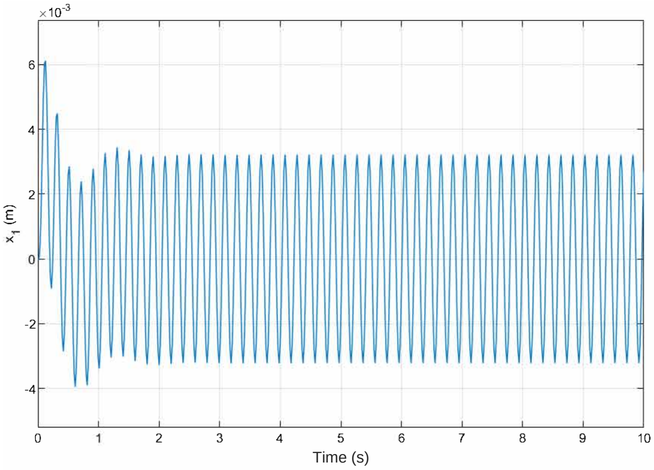


Figure 5. Simulink Response for $x_1$.

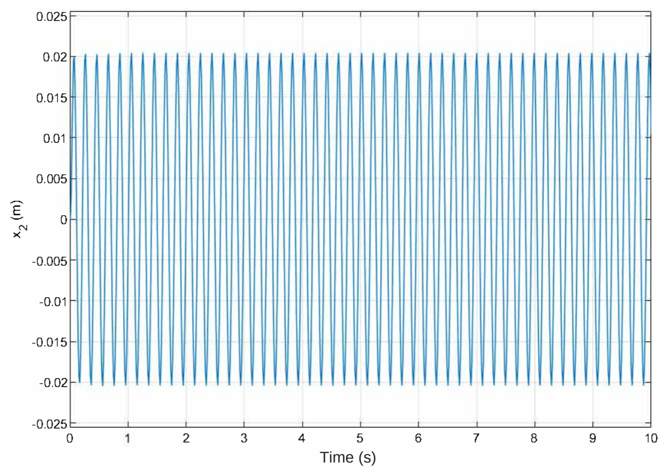


Figure 6. Simulink Response for $x_2$.

Therefore, comparing Figure 4 with Figure 5 and 6, I can know that the response derived by numeric method is almost the same as the that derived by Simulink method.

Next, I plot the numeric response of $x_1$ and $x_2$ without damper as shown in Figure 7. Comparing the diagram for car suspension system with and without damp, I can conclude that the damper function dissipates the energy of suspension vibrations where the damper absorbs a part of the shock energy directly when a vehicle goes over road excitation; it also guides spring action by dissipating the stored energy (Xie & Wang, 2015).

## 2.4 Trial for Different $k_1$ and $c$

At fist, let's conduct the analytical derivation of displacement of mass 1-$X_1$. From the analytical part in Section 2.2, I already get the solutions for 2-DoF damped system, which are shown in Equation 30. Substituting Equation 28 into Equation 31, I can obtain the displacement of mass 1, $X_1$, represented by $k_1$ and $c$. The determinant of mechanical impedance is equal to

$$\begin{aligned}
\det\left([Z(j\omega)]\right) &= \det\left(\begin{bmatrix} -\omega^2 m_{11} + j\omega c_{11} + k_{11} & -\omega^2 m_{12} + j\omega c_{12} + k_{12} \\ -\omega^2 m_{21} + j\omega c_{21} + k_{21} & -\omega^2 m_{22} + j\omega c_{22} + k_{22} \end{bmatrix}\right) \\
&= \left(-\omega^2 m_{11} + j\omega c_{11} + k_{11}\right)\left(-\omega^2 m_{22} + j\omega c_{22} + k_{22}\right) \\
&\quad - \left(-\omega^2 m_{12} + j\omega c_{12} + k_{12}\right)\left(-\omega^2 m_{21} + j\omega c_{21} + k_{21}\right) \\
&= \left(-\omega^2 m_{11} + k_{11}\right)\left(-\omega^2 m_{22} + k_{22}\right) - \left(-\omega^2 m_{12} + k_{12}\right)\left(-\omega^2 m_{21} + k_{21}\right) \\
&\quad + j\omega[c_{11}\left(-\omega^2 m_{22} + k_{22}\right) + c_{22}\left(-\omega^2 m_{11} + k_{11}\right) \\
&\quad - c_{12}\left(-\omega^2 m_{21} + k_{21}\right) - c_{21}\left(-\omega^2 m_{12} + k_{12}\right)] - \omega^2 c_{11} c_{22} + \omega^2 c_{12} c_{21}
\end{aligned} \tag{34}$$

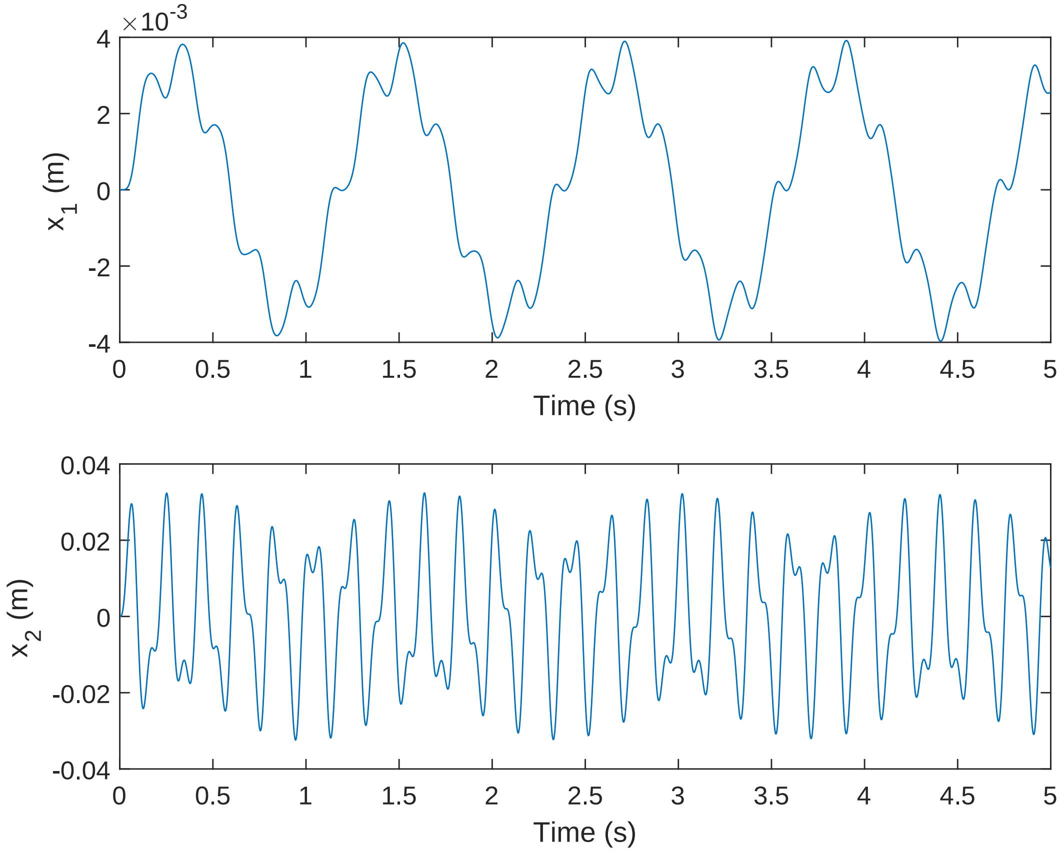


**Figure 7.** Numeric Response of Car Model without Damper.

Substituting the parameters of car suspension system model except $k_1$ and $c$ into Equation 34 yields that

$$\begin{aligned}
\det\left(\left[Z(j\omega)\right]\right) &= \left(-\omega^2 m_{11} + k_{11}\right)\left(-\omega^2 m_{22} + k_{22}\right) - \left(-\omega^2 m_{12} + k_{12}\right)\left(-\omega^2 m_{21} + k_{21}\right) \\
&\quad + j\omega[c_{11}\left(-\omega^2 m_{22} + k_{22}\right) + c_{22}\left(-\omega^2 m_{11} + k_{11}\right) \\
&\quad - c_{12}\left(-\omega^2 m_{21} + k_{21}\right) - c_{21}\left(-\omega^2 m_{12} + k_{12}\right)] - \omega^2 c_{11}c_{22} + \omega^2 c_{12}c_{21} \\
&= \left(-31.73^2 \times 1010 + k_1\right)\left[-31.73^2 \times 78 + (k_1 + 321100)\right] \\
&\quad - \left(-31.73^2 \times 0 - k_1\right)\left(-31.73^2 \times 0 - k_1\right) \\
&\quad + 31.73j\{c \times \left[-31.73^2 \times 78 + (k_1 + 321100)\right] \\
&\quad + c \times \left(-31.73^2 \times 1010 + k_1\right) \\
&\quad - (-c) \times \left(-31.73^2 \times 0 - k_1\right) - (-c) \times \left(-31.73^2 \times 0 - k_1\right)\} \\
&\quad - 31.73^2 \times c \times c + 31.73^2 \times (-c) \times (-c) \\
&= -2.487 \times 10^{11} - 7.725 \times 10^5 k_1 - 2.451 \times 10^7 cj
\end{aligned} \tag{35}$$

Therefore, I can find the displacement of mass 1 as show in Equation 36.

$$\begin{aligned} X_1 &= k_2 Y \cdot \frac{X_1}{F_{20}} = k_2 Y \cdot \left| \frac{-k_{12} + \omega^2 m_{12} - j\omega c_{12}}{\det\left(\left[Z(j\omega)\right]\right)} \right| \\ &= k_2 Y \cdot \left| \frac{-\left(-k_1\right) - 31.73\left(-c\right) j}{-2.487 \times 10^{11} - 7.725 \times 10^5 k_1 - 2.451 \times 10^7 cj} \right| \\ &= k_2 Y \frac{\sqrt{k_1^2 + (31.73c)^2}}{\sqrt{(-2.487 \times 10^{11} - 7.725 \times 10^5 k_1)^2 + (2.451 \times 10^7 c)^2}} \end{aligned} \tag{36}$$

In order to get the "best" suspension system, I need to the minimize $X_1$ in Equation 36.

$$\begin{aligned} X_1 &= k_2 Y \frac{\sqrt{k_1^2 + (31.73c)^2}}{\sqrt{(-2.487 \times 10^{11} - 7.725 \times 10^5 k_1)^2 + (2.451 \times 10^7 c)^2}} \\ &= \frac{k_2 Y}{\sqrt{5.968 \times 10^{11} + \frac{3.843 \times 10^{17} k_1}{k_1^2 + (31.73c)^2} + \frac{6.187 \times 10^{22}}{k_1^2 + (31.73c)^2}}} \end{aligned} \tag{37}$$

I use MATLAB to determine the characteristics of $X_1$ with respect to different $k_1$ and $c$, whose code is shown in Appendices F. The plot I obtain is shown in Figure 8.

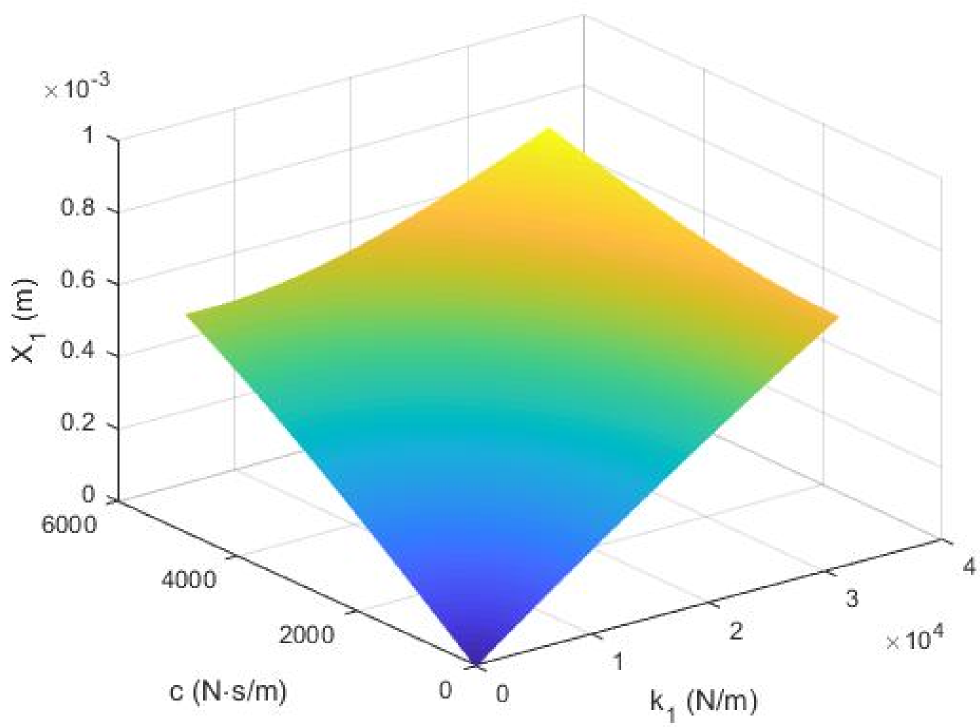


**Figure 8.** The Characteristics of $X_1$ with Respect to Different $k_1$ and $c$.

Hence, I can know that the minimum displacement $X_1$ occurs at $k_1 = 0$ and $c = 0$, which is unrealistic. In order to produce the "best" suspension system, I need to choose appropriate value for $k_1$ and $c$. In this model, I will choose $k_1 = 5000$ and $c = 1500$ as my parameter. And I use the code shown in Appendices G to plot the response with $k_1 = 5000$ and $c = 1500$ as shown in Figure 9.

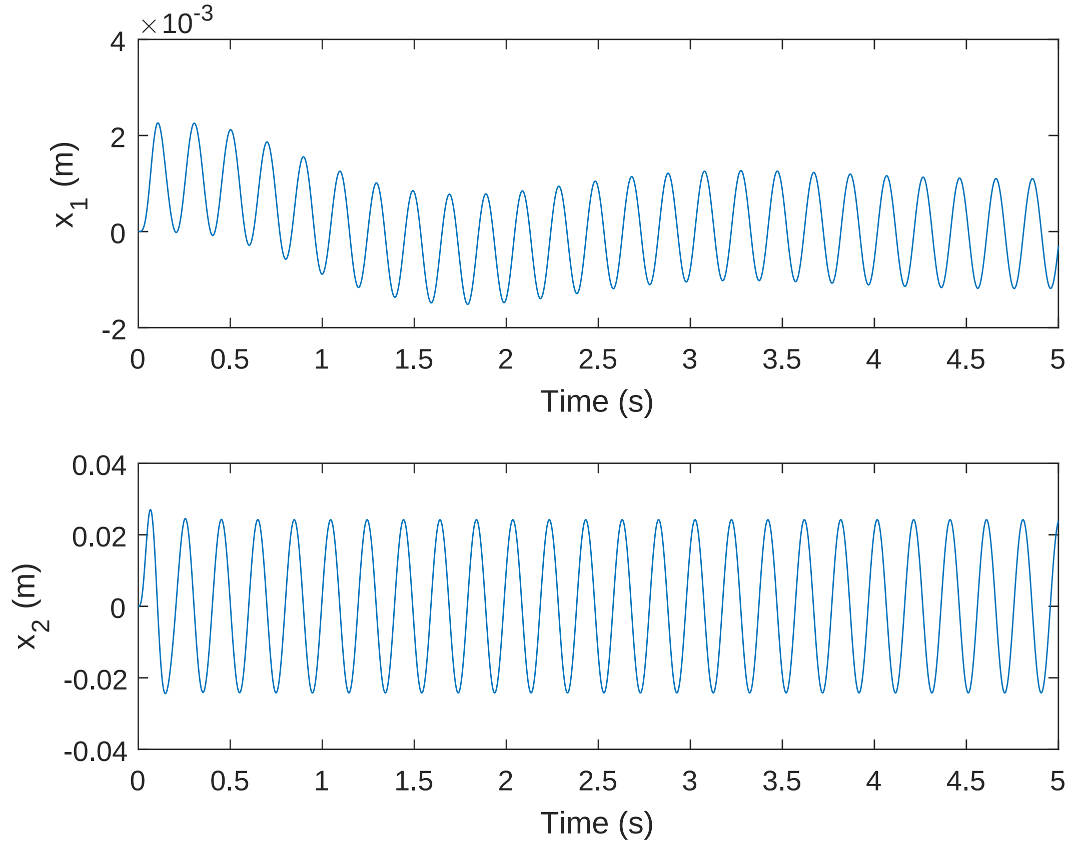


**Figure 9.** The Response for the Car Suspension System when $k_1 = 5000$ and $c = 1500$.

# 3 Discussion Part: Other Means of Vibration Control for the Car Suspension System

There are three methods of suspension system strategies; passive system, active system, and semi–active system. The passive suspension system is the model which we have analyzed in Section 2. Although the passive suspension is still utilized in vehicle industry, these requirements cannot be achieved by a passive suspension with steel springs and passive dampers. Therefore, an increasing need has appeared over the last years for new suspension systems to maintain the level of comfort that customers expect from cars, they still maintain high safety standards of vehicles while taking into account the cost reduction (Moheyeldein et al., 2018). The semi-active control included; magneto rheological damper and electro rheological damper, in this case; the effective area in which the oil damping flowing through was varied according to road disturbance (Sam et al., 2004). An active suspension is a type of automotive suspension on a vehicle. It uses an onboard system to control the vertical movement of the vehicle's wheels relative to the chassis or vehicle body rather than the passive suspension provided by large springs where the movement is determined entirely by the road surface (Van der Sande et al., 2013). So-called active suspensions are divided into two classes: real active suspensions, and adaptive or semi-active suspensions. While adaptive suspensions only vary shock absorber firmness to match changing road or dynamic conditions, active suspensions use some type of actuator to raise and lower the chassis independently at each wheel (Lin & Lian, 2010).

### 3.1 Passive Suspension System

Passive suspension systems comprise springs and dampers inserted between the body of vehicle and the wheel-axle assembly. Passive suspensions have the advantages of simple mechanism, easy implementation and high reliability, but they are inadequate in improving ride comfort or road holding for the reason that invariant spring and damper characteristics are unable to cope with different road conditions and conflicting criteria (Tamboli & Joshi, 1999) (Sharp & Hassan, 1986) (Naudé & Snyman, 2003). Because it has already been discussed in Section 2, I will not go through in detail in this section.

### 3.2 Semi–Active Suspension System

Semi-active suspension systems feature variable dampers or springs, which means that the damping coefficients or the spring stiffness can be adjusted within a given range. Due to their low energy consumption and high reliability, they are available in a wide range of production vehicles (Du et al., 2005) (Paulides et al., 2006). However, the resulting damper forces or spring forces are restricted by passivity constraints, i.e., they can only counteract the relative motion of the damper and dissipate energy passively, which are limited in improving ride comfort although they represent a considerable improvement over passive suspension systems.

### 3.3 Active Suspension System

An active control system, as shown in Figure 10, required external power source or many control actuators that apply forces to the suspension in prescribed manner. These forces can be required to both add and dissipate energy in the suspension (Sun et al., 2020). The components of each suspension system is shown in Figure 11. However, due to their energy requirements as well as weight and packaging aspects, active suspension systems have not been integrated in production vehicles, but undoubtedly, active suspensions will be the trend of future vehicle suspension design (Li et al., 2011).

## 4 Conclusion

In this project, I have explored simple passive car suspension system through their analytical and numerical also Simulink response due to the irregularity of the road, determined the proprieties, i.e. displacement transmissibility, force transmissibility, natural frequencies and mode shapes, of the model, and reanalyzed the suspension system without damper to figure out the role of damper in car suspension system. Besides, I have also searched literature for other means of vibration control for the suspension system and found the characteristics of each method. After going through this project, I have a more in-depth understanding of how the car suspension system works, how the car uses the suspension system to increase comfort and safety, and the principles of modern advanced car suspension systems, which has laid a solid foundation for me to engage in automobile research in the future.

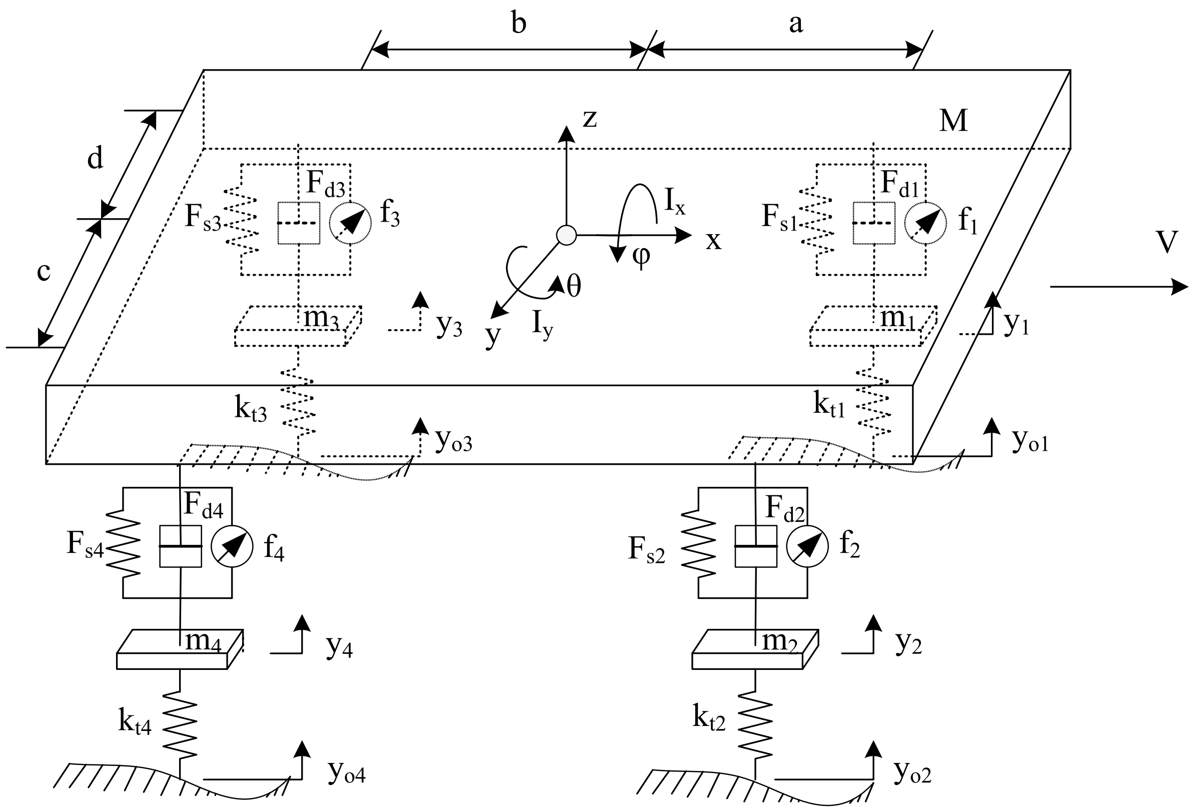


**Figure 10.** Sketch for Active Suspension System.

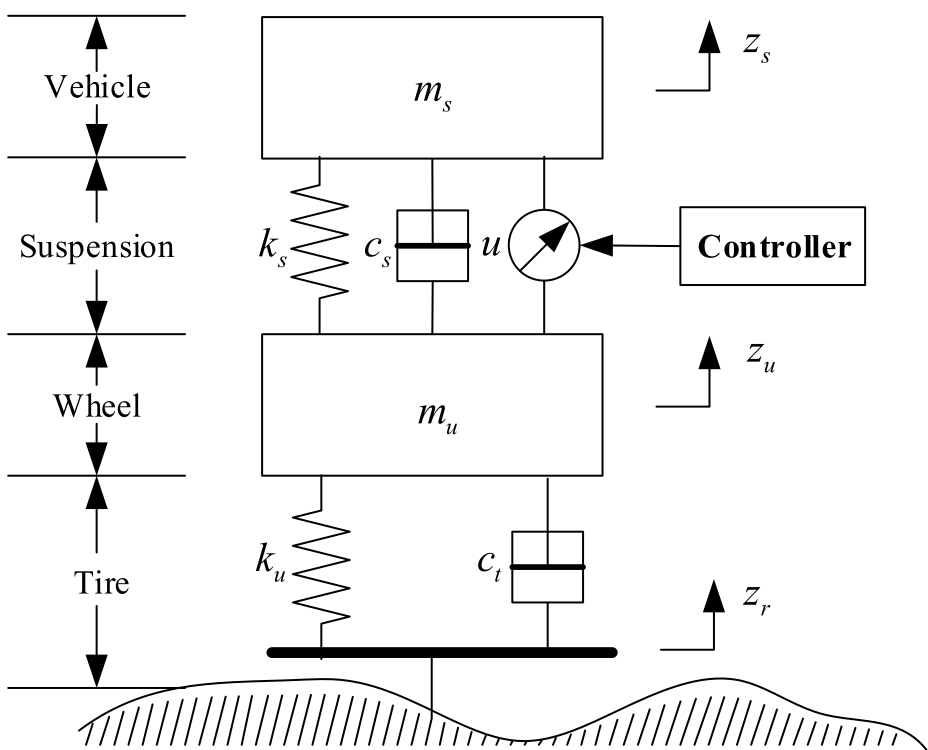


**Figure 11.** Components of Each Suspension System.

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# Appendices

## A Natural Frequencies and Mode Shapes Code

**Input Matlab source for Calculation of Natural Frequencies and Mode Shapes:**

```
clc; clf; clear all;
% Clear all the things that are remained in the before command.
m1 = 1010;
m2 = 76;
k1 = 31110;
k2 = 321100;
% Properties of the system.
m = [m1 0; 0 m2];
k = [k1 -k1; -k1 k1+k2];
% Define [M] and [K] matrices
eigsort(k,m);
function [u,wn] = eigsort(k,m)
    Omega = sqrt(eig(k,m));
    [vtem,~] = eig(k,m);
    % Obtain eigenvalues and eigenvectors of A
    [wn,isort] = sort(Omega);
    % Determine the natural frequencies of the 2 DoF system.
    wnlength = length(wn);
    for i = 1:wnlength
        v(:,i) = vtem(:,isort(i));
    end
    % Determine the eigenvectors of the system.
    A1_A2_1 = v(1,1)/v(2,1);
    A1_A2_2 = v(1,2)/v(2,2);
    % Determine the ratios of eigenvectors.
    disp("The natural frequencies are (rad/sec)")
    disp(wn)
    disp("The eigenvectors of the system are")
    disp(v)
    disp("Ratios of eigenvectors are:")
    disp(A1_A2_1);
    disp(A1_A2_2);
    % Display the parameters I want.
    figure(1);
    % Open one figure.
    plot([0,1,2,3], [0,A1_A2_1,1,0],'b-s', 'LineWidth', ...
        2, 'MarkerSize',10);
    hold on;
    plot([0,1,2,3], [0,-A1_A2_2,-1,0],'r-s', 'LineWidth', ...
        2, 'MarkerSize',10);
    hold off;
    xlabel('Mass Number');
    ylabel('Eigne Vector Ratio');
    xticks([1 2]);
    legend('Mode 1', 'Mode 2')
    grid on;
```

```
    % Plot the mode shape of the car model.
end
```

# B Calculation of Transfer Function for the System

**Input Matlab source for Calculation of Natural Frequencies and Mode Shapes:**

```
clc; clf; clear all;
% Clear all the things that are remained in the before command.
m1 = 1010;
m2 = 76;
c = 4850;
k1 = 31110;
k2 = 321100;
% Properties of the system.
A = [0 0 1 0;
    0 0 0 1;
    -k1/m1 k1/m1 -c/m1 c/m1;
    k1/m2 -(k1+k2)/m2 c/m2 -c/m2];
B = [0 0; 0 0; 0 0; 0 k2/m2];
C = [1 0 0 0; 0 1 0 0];
D = zeros(2,2);
% State-space representation matrix of the system.
sys_ss = ss(A,B,C,D);    % Get the state-space representation.
sys_tf = tf(sys_ss);     % Get the transfer function.
```

# C Calculation of Displacement Transmissibility and Force Transmissibility

**Input Matlab source for Displacement Transmissibility and Force Transmissibility:**

```
clc; clf; clear all;
% Clear all the things that are remained in the before command.
m1 = 1010;
m2 = 76;
c = 4850;
k1 = 31110;
k2 = 321100;
omega = 100/3.6/5.5*2*pi;
% Properties of the system.
M = [m1 0; 0 m2];
C = [c -c; -c c];
K = [k1 -k1; -k1 k1+k2];
% Define [M], [C] and [K] matrices.
Z = -omega^2*M+1i*omega*C+K;
% Find the mechanical impedance.
% X_k2Y = zeros(2,2);
% X_k2Y(1,1) = abs((K(1,1)-omega^2*M(1,1)+1i*omega*C(1,1))/(det(Z)));
```

```
% X_k2Y(1,2) = abs((-K(1,2)+omega^2*M(1,2)-1i*omega*C(1,2))/(det(Z)));
% X_k2Y(2,1) = abs((-K(2,1)+omega^2*M(2,1)-1i*omega*C(2,1))/(det(Z)));
% X_k2Y(2,2) = abs((K(2,2)-omega^2*M(2,2)+1i*omega*C(2,2))/(det(Z)));
X_k2Y = abs((K-omega^2*M+1i*omega*C)/(det(Z)));
% Find the matrix of X_i/F_i.
X1_Y = k2*X_Y(1,2);
X2_Y = k2*X_Y(1,1);
fprintf('Displacement Transmissibility for mass 1 = %.3f. \n', X1_Y);
fprintf('Displacement Transmissibility for mass 2 = %.3f. \n', X2_Y);
% Find the displacement transmissibility.
F1_k2Y = abs(k1*(X_k2Y(1,2)-X_k2Y(1,1)));
F2_k2Y = abs(k2*(X_k2Y(1,1)-1/k2)-k1*(X_k2Y(1,2)-X_k2Y(1,1)));
fprintf('Force Transmissibility for mass 1 = %.3f. \n', F1_k2Y);
fprintf('Force Transmissibility for mass 2 = %.3f. \n', F2_k2Y);
% Find the force transmissibility.
```

# D Code for Numeric Solution for Car Motion

## D.1 Code in Numeric Response with Damper Main File

**Input Matlab source for Main File:**

```
% Main program (save this file as Mainchristopher.m)
clc; clf; clear all;
% Clear all the things that are remained in the before command.
global m1 m2 k1 k2 cc m k c              % Set m, k, c as global variance.
m1 = 1010;
m2 = 76;
k1 = 31110;
k2 = 321100;
cc = 4850;
% Properties of the system.
m = [m1 0; 0 m2];          % Set the mass matrix.
c = [cc -cc; -cc cc]; % Set the damping matrix.
k = [k1 -k1; -k1 k1+k2];       % Set the stiffness matrix.
z0 = [0; 0; 0; 0];
% Set the initial conditions for [x1, x2, x1dot, x2dot]
tspan = [0:0.001:5];      % Set the t range.
[t, z] = ode45('christopher', tspan, z0);
% Solve the response.
subplot(211);         % Open a subplot.
plot(t, z(: , 1));  % plot z1
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_1 (m)');  % Set the y label to x_1 (m).
subplot(212);         % Open a subplot.
plot(t, z(: , 2));  % plot z2
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_2 (m)');  % Set the y label to x_2 (m).
```

## D.2 Code in Numeric Response without Damper Main File

**Input Matlab source for Main File:**

```
% Main program (save this file as Mainchristopher.m)
clc; clf; clear all;
% Clear all the things that are remained in the before command.
global m1 m2 k1 k2 cc m k c              % Set m, k, c as global variance.
m1 = 1010;
m2 = 76;
k1 = 31110;
k2 = 321100;
cc = 0;
% Properties of the system.
m = [m1 0; 0 m2];          % Set the mass matrix.
c = [cc -cc; -cc cc]; % Set the damping matrix.
k = [k1 -k1; -k1 k1+k2];       % Set the stiffness matrix.
z0 = [0; 0; 0; 0];
% Set the initial conditions for [x1, x2, x1dot, x2dot]
tspan = [0:0.001:5];      % Set the t range.
[t, z] = ode45('christopher', tspan, z0);
% Solve the response.
subplot(211);         % Open a subplot.
plot(t, z(: , 1));  % plot z1
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_1 (m)');  % Set the y label to x_1 (m).
subplot(212);         % Open a subplot.
plot(t, z(: , 2));  % plot z2
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_2 (m)');  % Set the y label to x_2 (m).
```

## D.3 Code in Numeric Response Function File

**Input Matlab source for Function File:**

```
function zdot = christopher(t, z)
    global m k c k2
    % Set m, k, c as global variance.
    [nr, nc] = size(m);
    % Find the size of m.
    A = [zeros(nr, nr) eye(nr); -inv(m)*k -inv(m)*c];
    B = [zeros(nr, nr); inv(m)];
    F = [0; k2*0.019*sin(100/3.6/5.5*2*pi*t)];
    zdot = A*z + B*F;
```

# E Simulink Solution for Car Motion

## E.1 Simulink Block Diagram

The block diagram for the car motion I simulinked is shown in Figure 12. The code in the MATLAB Function Block is shown in Appendices E.2 and E.3

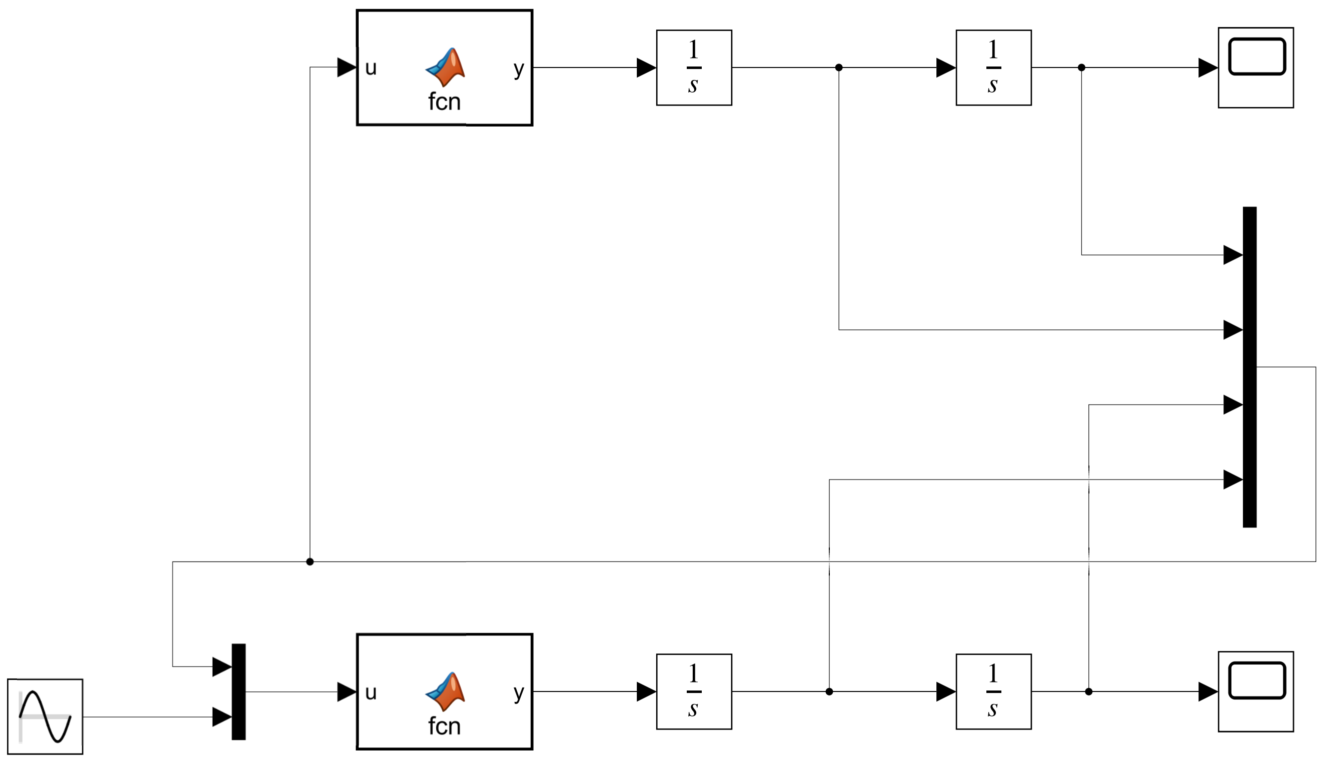


**Figure 12.** Block Diagram for the Car Motion.

## E.2 Code in Upper Function Block

**Input Matlab source for Above Function Block (Fcn):**

```
function y = fcn(u)
    m1 = 1010;
    m2 = 76;
    k1 = 31110;
    k2 = 321100;
    c = 4850;
    % Set the properties of the system.
    y = -(c*u(2)-c*u(4)+k1*u(1)-k1*u(3))/m1;
    % Get the expression for x1ddot.
```

### E.3 Code in Lower Function Block

**Input Matlab source for Below Function Block (Fcn):**

```
function y = fcn(u)
    m1 = 1010;
    m2 = 76;
    k1 = 31110;
    k2 = 321100;
    c = 4850;
    % Set the properties of the system.
    y = (k2*u(5)-(-c*u(2)+c*u(4)-k1*u(1)+(k1+k2)*u(3)))/m2;
    % Get the expression for x1ddot.
```

## F Code for Minimum Displacement

**Input Matlab source to find the characteristics of $X_1$ with different $k_1$ and $c$:**

```
clc; clf; clear all;
% Clear all the things that are remained in the before command.
k1range = 0:100:31110;
crange = 0:5:4850;
% Set the range of k1 and c.
[k1, c] = meshgrid(k1range, crange);
% Generate the matrix of k1 and c.
X1 = 321100*0.019*sqrt(k1.^2+(31.73*c.^2))./...
    sqrt((-2.487e11-7.725e5*k1).^2+(2.451e7*c).^2);
% Define the displacement x1.
figure;
% Open one figure.
mesh(k1, c, X1);
% Plot X1 with respect to k1 and c.
xlabel('k_1 (N/m)');
ylabel('c (N\cdots/m)');
zlabel('X_1 (m)');
% Set the label of the plot.
```

## G Code for Appropriate $k_1$ and $c$

**Input Matlab source to find the response with appropriate $k_1$ and $c$:**

```
% Main program (save this file as Mainchristopher.m)
clc; clf; clear all;
% Clear all the things that are remained in the before command.
global m1 m2 k1 k2 cc m k c             % Set m, k, c as global variance.
m1 = 1010;
m2 = 76;
k1 = 5000;
k2 = 321100;
```

```
cc = 1500;
% Properties of the system.
m = [m1 0; 0 m2];          % Set the mass matrix.
c = [cc -cc; -cc cc]; % Set the damping matrix.
k = [k1 -k1; -k1 k1+k2];       % Set the stiffness matrix.
z0 = [0; 0; 0; 0];
% Set the initial conditions for [x1, x2, x1dot, x2dot]
tspan = [0:0.001:5];     % Set the t range.
[t, z] = ode45('christopher', tspan, z0);
% Solve the response.
subplot(211);        % Open a subplot.
plot(t, z(: , 1));  % plot z1
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_1 (m)');  % Set the y label to x_1 (m).
subplot(212);        % Open a subplot.
plot(t, z(: , 2));  % plot z2
xlabel('Time (s)'); % Set the x label to Time (s).
ylabel('x_2 (m)');  % Set the y label to x_2 (m).
```