ME1020
Mechanical vibrations

Lecture 6

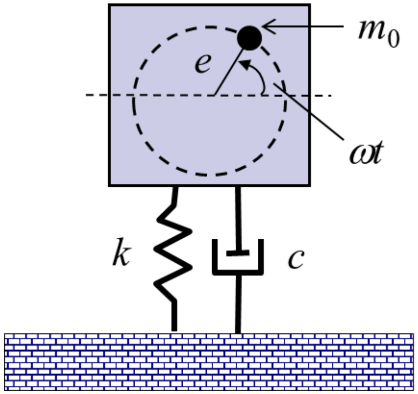
Rotating imbalance

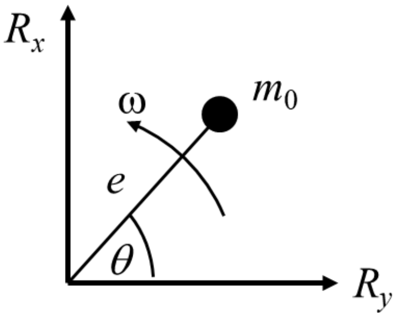
Objectives

- Analyze the response of 1DOF vibration system to rotating imbalanced mass including the amplitude ratio, phase shift, and force transmission
- Apply the rotating imbalanced mass analysis to vibration isolation
- Describe the characteristics of whirling including steady state amplitudes and critical speed

Introduction

- Rotating machines are not always perfectly balanced, e.g. your car's wheels
- Rotating machine imbalance can be due to manufacturing or wear (e.g. turbine engine with cracked turbine blades)
- Rotating imbalance leads to harmonic forces and is one of the main causes of machine vibration
- A simple model of a system with an unbalance mass rotating at constant angular velocity ω at a radius "e" is shown





Introduction

Let m = total mass of machine (including m_0)

- m_0 = mass imbalanced
- e = eccentricity

  ω = rotation frequency

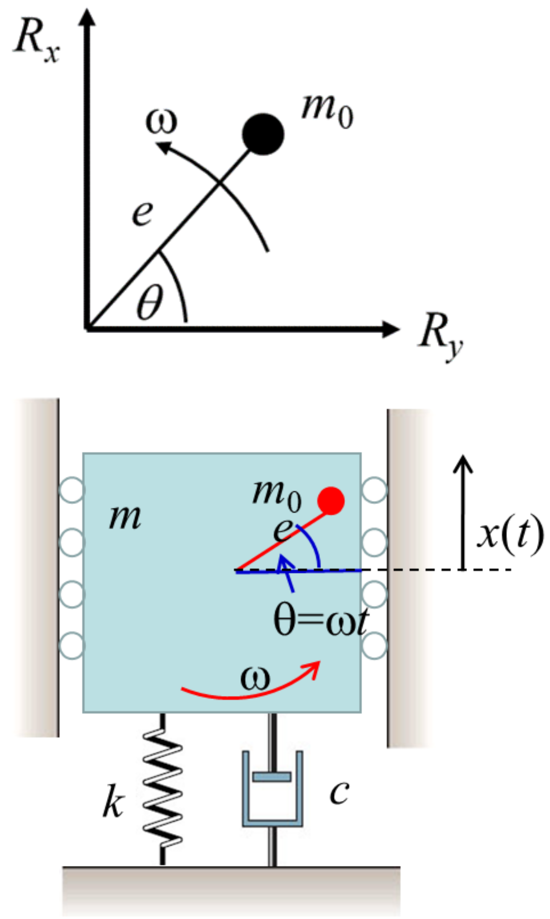
Note that $x = e\sin(\omega t)$ and $\ddot{x} = -e\omega^2\sin(\omega t)$

$$R_x = m_0\ddot{x} = -m_0 e\,\omega^2\sin(\omega t)$$

$$R_y = m_0\ddot{y} = -m_0 e\,\omega^2\cos(\omega t)$$

- Assume that the mass is held in place along y-axis
- The force acting against the mass will be $-R_x$
- The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = m_0 e\,\omega^2\sin(\omega t)$$

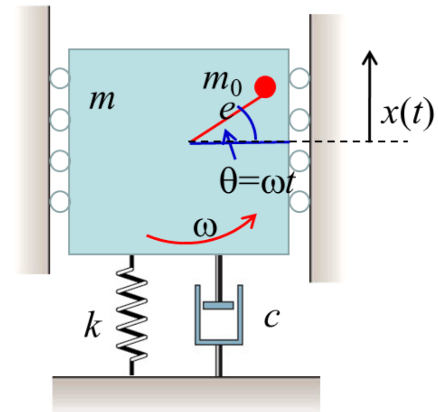


Rotating imbalance

For harmonic force excitation: $m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$

❖ Steady state response is $x(t) = X \sin(\omega t - \theta)$

where $X = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}}$ and $\theta = \tan^{-1} \frac{c\omega}{[k - m\omega^2]}$



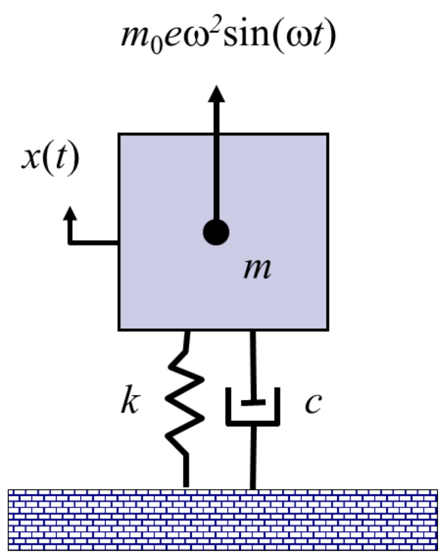
For a rotating imbalance mass:

$$m\ddot{x} + c\dot{x} + kx = m_0 e\, \omega^2 \sin(\omega t)$$

❖ Steady state response is $x(t) = X \sin(\omega t - \theta)$

where $X = \frac{m_0 e\, \omega^2}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}} = \frac{r^2 (m_0 e)/m}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$

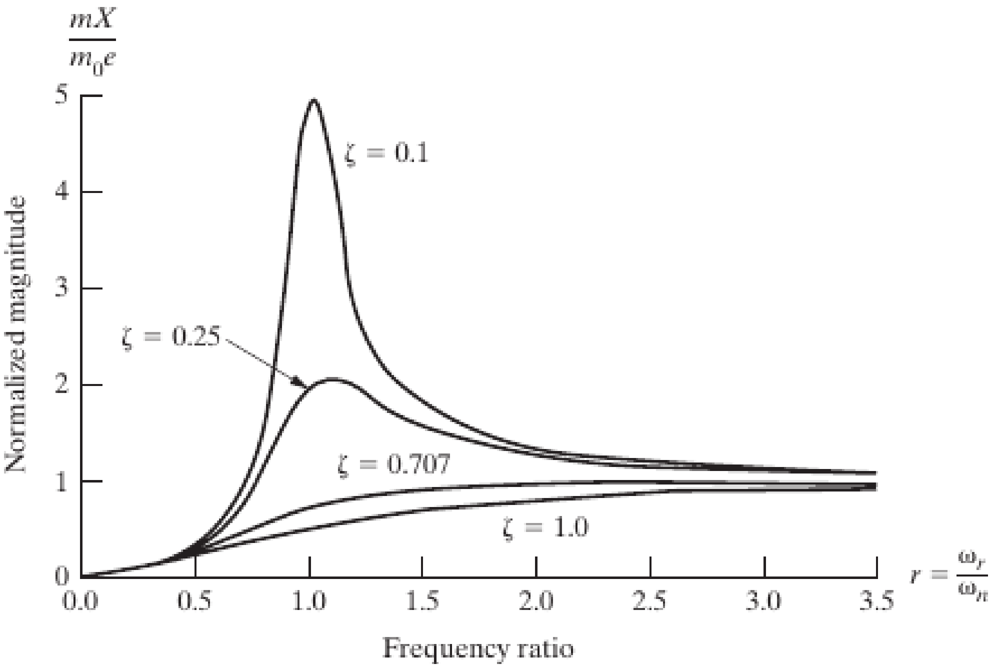
Phase shift $\theta = \tan^{-1} \frac{c\omega}{[k - m\omega^2]} = \tan^{-1} \frac{2\zeta r}{[1 - r^2]}$



Amplitude ratio

The amplitude ratio for a rotating imbalance system is defined as

$$\frac{X}{(m_0 e)/m} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



Amplitude ratio

The amplitude ratio shows the following characteristics:

1. At very high speeds ($\omega \to$ large $\Longrightarrow r \to$ large), $mX/m_0e \to 1$ and the effect of damping is negligible

2. For $0 < \zeta < 1/\sqrt{2}$, the maximum of mX/m_0e occurs at

$$\frac{d}{dr}\left(\frac{mX}{m_0e}\right) = 0, \text{ i.e. at } \ r = \frac{1}{\sqrt{1-2\zeta^2}}$$

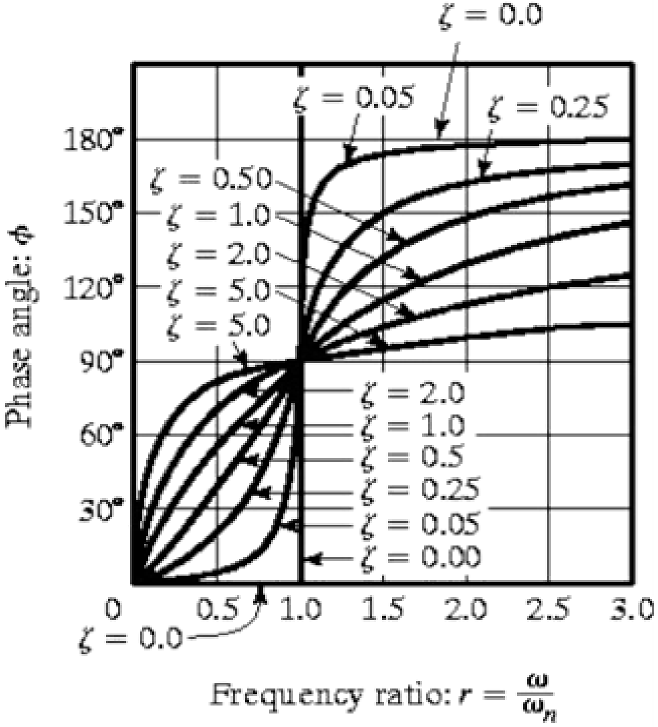
   The maximum occurs to the right of the resonance value of $r = 1$

3. For $\zeta > 1/\sqrt{2}$, the amplitude ratio does not attain a maximum but grows from 0 at $r = 0$ to 1 at $r \to \infty$

Phase shift

The phase shift for a rotating imbalance system is

$$\theta = \tan^{-1}\frac{2\zeta r}{[1 - r^2]}$$



Example 1

A machine has a rotating imbalance. At resonance, the maximum deflection is measured to be 0.1 m. The damping ratio is estimated to be 0.05 and the out-of-balance mass, m_0, is estimated to be 10% of the total mass. Estimate the eccentricity radius e and determine how much mass should be added (uniformly) to the system to reduce the deflection at resonance to 0.01 m.

Example 1

At resonance $r = 1$

$$X = \frac{r^2\,(m_0 e)/m}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{(m_0 e)/m}{2\zeta}$$

Given $\zeta = 0.05$, $m_0 = 0.1m$ and measured deflection at resonance to 0.1 m.

Therefore

$$0.1 = \frac{(0.1me)/m}{2\zeta}$$

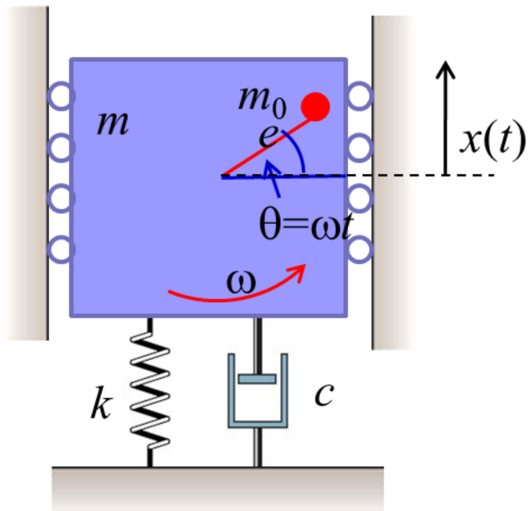
$$e = 2\zeta = 0.1 \text{ m;}$$

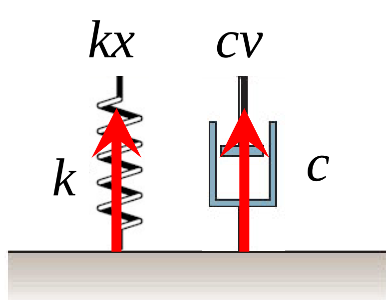
Desirable to change mass to $m + \Delta m$ so that $X = 0.01$ m; i.e.

$$0.01 = \frac{(0.1me)/(m+\Delta m)}{2\zeta} \quad \text{or} \quad m + \Delta m = 10m$$

$$\Delta m = 9m \text{ (i.e. increase mass by 9 times)}$$

Force transmission





- With vibrating machinery, forces exerted on the supporting structures can become large near resonance
- Equipment is thus constructed on isolating mounts (springs and dashpots to suppress the resonance)
- Steady state displacement is

$$x(t) = X\sin(\omega t - \theta)$$

- Force transmitted is

$$F_T = kx + c\dot{x}$$

$$F_T = kX\sin(\omega t - \theta) + c\omega X\cos(\omega t - \theta)$$

Force transmission

Force transmitted is

$$F_T = kX\sin(\omega t - \theta) + c\omega X\cos(\omega t - \theta)$$

- The sine and cosine can be combined and the maximum amplitude of the transmitted force is

$$|F_T| = \sqrt{(kX)^2 + (c\omega X)^2} = kX\sqrt{1 + (2\zeta\omega/\omega_n)^2}$$

- Substitute $X = \frac{r^2(m_0e)/m}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

$$|F_T| = \frac{r^2\,(m_0e)(k/m)\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$|F_T| = \frac{m_0e\omega^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Vibration isolation

The force transmitted from the rotating unbalanced mass through the isolator consisting of the spring and damper is

$$|F_T| = \frac{m_0 e\omega^2 \sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

The force transmissibility of the isolator is defined as

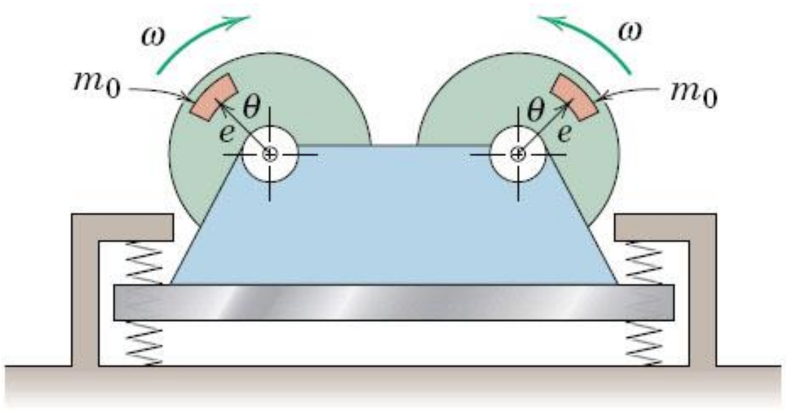
$$T_r = \frac{F_T}{m_0 e\omega^2} = \sqrt{\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}}$$

The reduction in force transmissibility is defined as

$$R = 1 - T_r$$

Example 2

Determine the two possible values of the equivalent spring stiffness k for the mounting to permit the amplitude of the force transmitted to the fixed mounting due to the imbalance to be 1500 N at a speed of 1800 rpm. Given: total mass of the device $m = 10$ kg, unbalanced mass $m_0 = 1$ kg; and eccentricity $e = 12$ mm. Assume negligible damping



Example 2

Note: negligible damping and $\zeta = 0$,

Force transmitted is 1500 N at a speed of 1800 rpm;

But force transmitted is $|F_T| = \frac{m_0 e\omega^2 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{m_0 e\omega^2}{|1-r^2|}$

Therefore $|1 - r^2| = \frac{m_0 e\omega^2}{1500}$

- Forcing frequency $\omega = 1800 \times 2\pi \div 60$ rad/s
- Eccentricity $e = 12$ mm and total unbalanced mass $m_0 = 2$ kg

$$|1 - r^2| = \frac{m_0 e\omega^2}{1500} = 0.568$$

$$r^2 - 1 = 0.568 \quad \text{or} \quad 1 - r^2 = 0.568$$

$$r^2 = 1.568 \quad \text{or} \quad r^2 = 0.432$$

- Natural frequency $\omega_n^2 = k/m$ and $r^2 = \omega^2/\omega_n^2 = m\omega^2/k$
- $m\omega^2/k = 1.568$ gives $k = m\omega^2/1.568 = 2.27 \times 10^5$ N/m
- $m\omega^2/k = 0.432$ gives $k = m\omega^2/0.432 = 8.23 \times 10^5$ N/m

Example 2

Note: Device total mass $m = 10$ kg

- ❖ Forcing frequency $\omega = 188.5$ rad/s
- ❖ If we choose the softer spring $k = 2.27 \times 10^5$ N/m

  Natural frequency $\omega_n = \sqrt{k/m} = 150.7$ rad/s; and

  $r = \omega/\omega_n = 1.25$

- ❖ If we choose the stiffer spring $k = 8.23 \times 10^5$ N/m

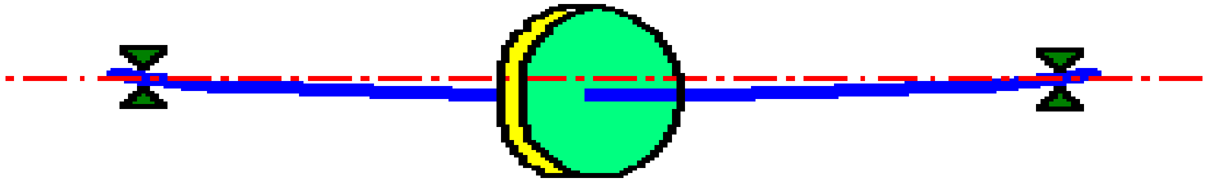
  Natural frequency $\omega_n = \sqrt{k/m} = 286.9$ rad/s; and

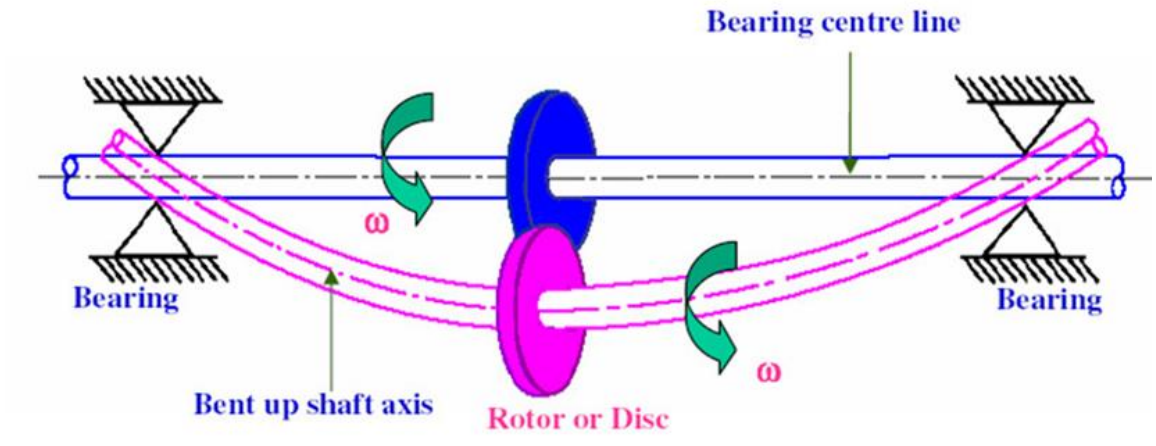
  $r = \omega/\omega_n = 0.657$

- ❖ We can choose either a stiffer spring and run the machine BELOW the natural frequency or we can choose a softer spring and run the machine ABOVE the natural frequency

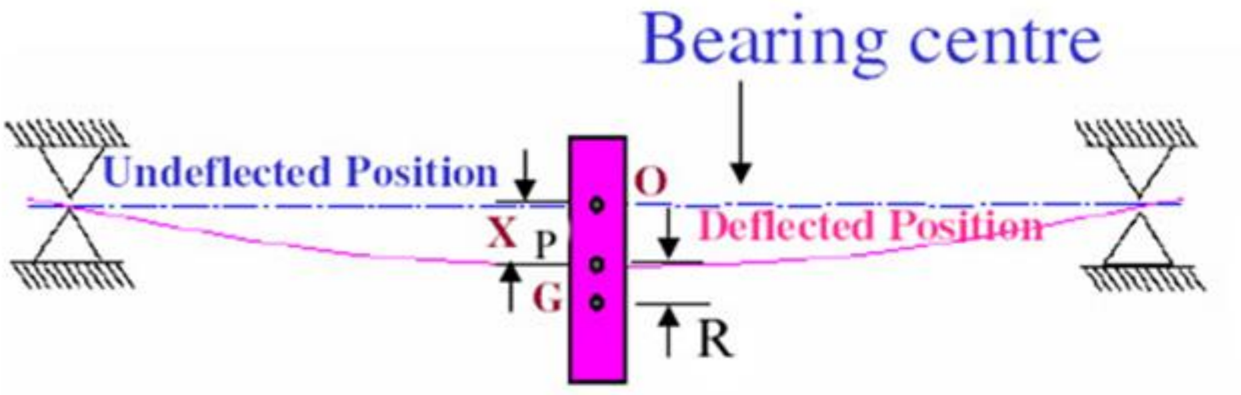
Whirling

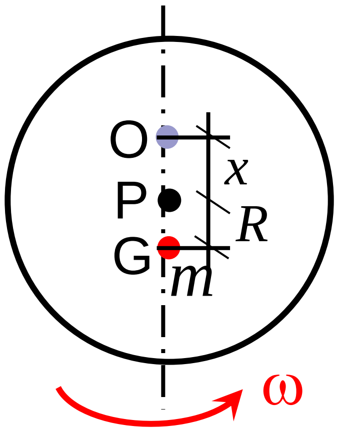
In machines with rotating shaft, it is possible that the rotating shaft can bend. The resulting motion of the off-center mass is called whirling





Whirling



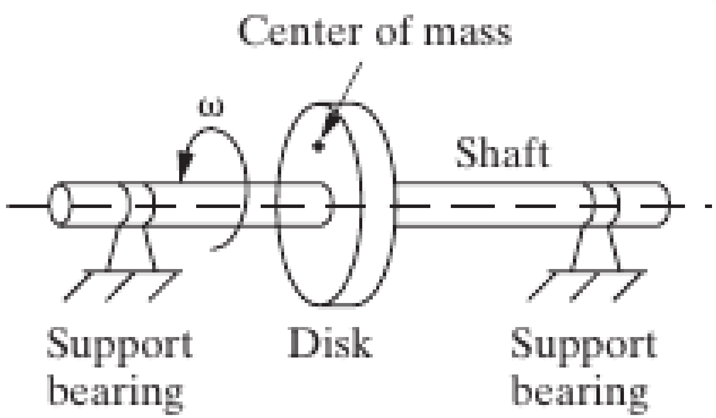


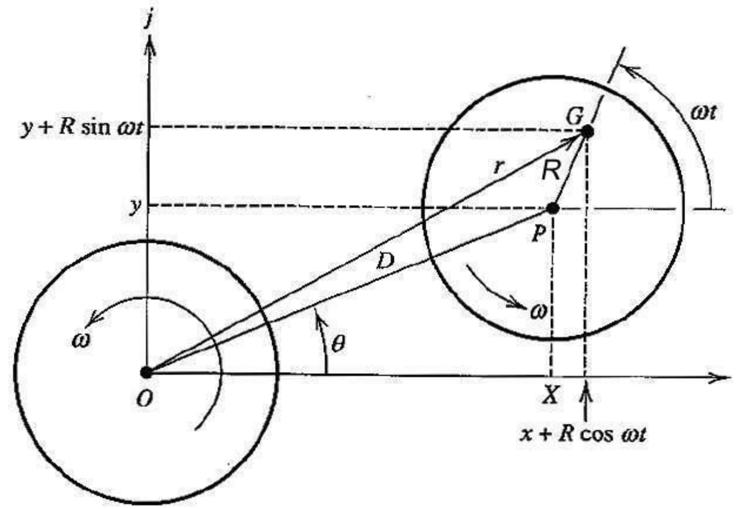
Terminology:

- O - center of rotation (of unbend shaft)
- P - Geometric center (center of rotation of bend shaft)
- x - deflection of shaft
- G - center of gravity where mass m is located
- R – eccentricity

Note: For perfect balancing, the center of gravity has to coincide with the geometric center

Whirling





Force balance: $m\ddot{\vec{r}} = -kx\hat{\imath} - ky\hat{\jmath} - c\dot{x}\hat{\imath} - c\dot{y}\hat{\jmath}$

$$\vec{r} = (x + R\cos\omega\, t)\hat{\imath} + (y + R\sin\omega\, t)\hat{\jmath}$$

$$\ddot{\vec{r}} = (\ddot{x} - R\omega^2\cos\omega\, t)\hat{\imath} + (\ddot{y} - R\omega^2\sin\omega\, t)\hat{\jmath}$$

$$(m\ddot{x} - mR\omega^2\cos\omega\, t + c\dot{x} + kx)\hat{\imath} + (m\ddot{y} - mR\omega^2\sin\omega\, t + c\dot{y} + ky)\hat{\jmath} = 0$$

Note: m = mass of the rotor; r = position vector from O to CG; R = eccentricity; k = bending stiffness of the shaft; and c = external damping of the shaft

Whirling

Equation of motion of the whirling shaft:

$$(m\ddot{x} - mR\omega^2 \cos\omega\, t + c\dot{x} + kx)\hat{\imath} + (m\ddot{y} - mR\omega^2 \sin\omega\, t + c\dot{y} + ky)\hat{\jmath} = 0$$

Note: there are 2 equations:

$$m\ddot{x} - mR\omega^2 \cos\omega\, t + c\dot{x} + kx = 0$$
$$m\ddot{y} - mR\omega^2 \sin\omega\, t + c\dot{y} + ky = 0$$

- ❖ These equations can be rewritten as

$$m\ddot{x} + c\dot{x} + kx = mR\omega^2 \cos\omega\, t$$
$$m\ddot{y} + c\dot{y} + ky = mR\omega^2 \sin\omega\, t$$

- ❖ The steady state solutions for these are respectively

$$x(t) = X\cos(\omega t - \phi)$$
$$y(t) = Y\sin(\omega t - \phi)$$

where

$$X = Y = \frac{R(r)^2}{\sqrt{[1-r^2]^2 + [2\varsigma r]^2}} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{2\varsigma r}{1-r^2}\right)$$

Whirling

Note that θ in the diagram is given by

$$\tan\theta = \frac{y(t)}{x(t)} = \frac{\sin(\omega t - \phi)}{\cos(\omega t - \phi)}$$

$$\tan\theta = \tan(\omega t - \phi)$$

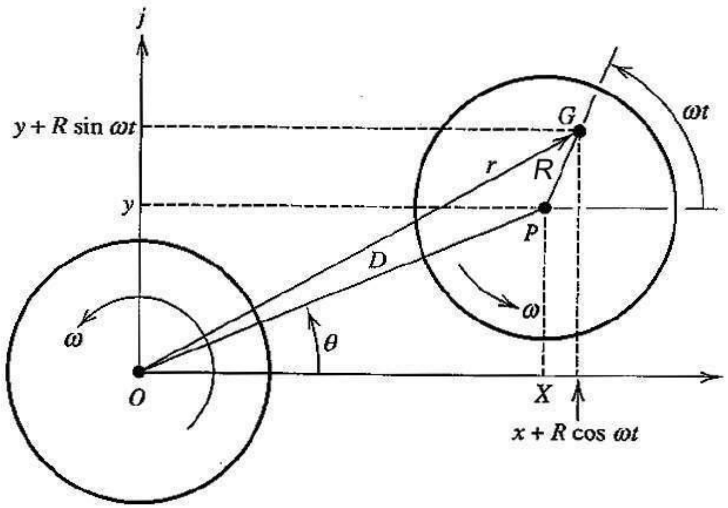
$$\theta = \omega t - \phi$$

$$\dot{\theta} = \omega$$

- ❖ The whirling velocity is the same as the speed with which the disk rotates about the shaft. This is called synchronous whirl
- ❖ In the diagram, distance "OP" is

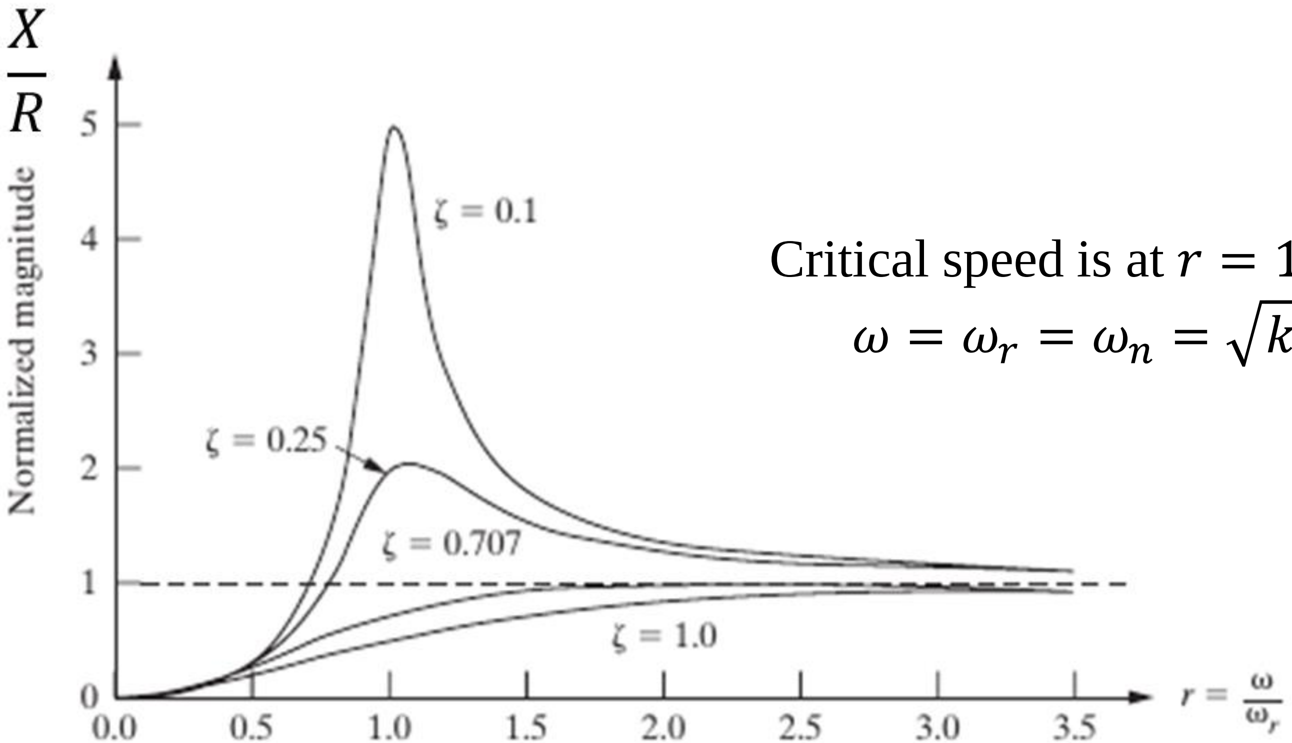
$$D = \sqrt{X^2 + Y^2} \quad \text{where}$$

$$X = Y = \frac{R(r)^2}{\sqrt{[1 - r^2]^2 + [2\varsigma r]^2}}$$



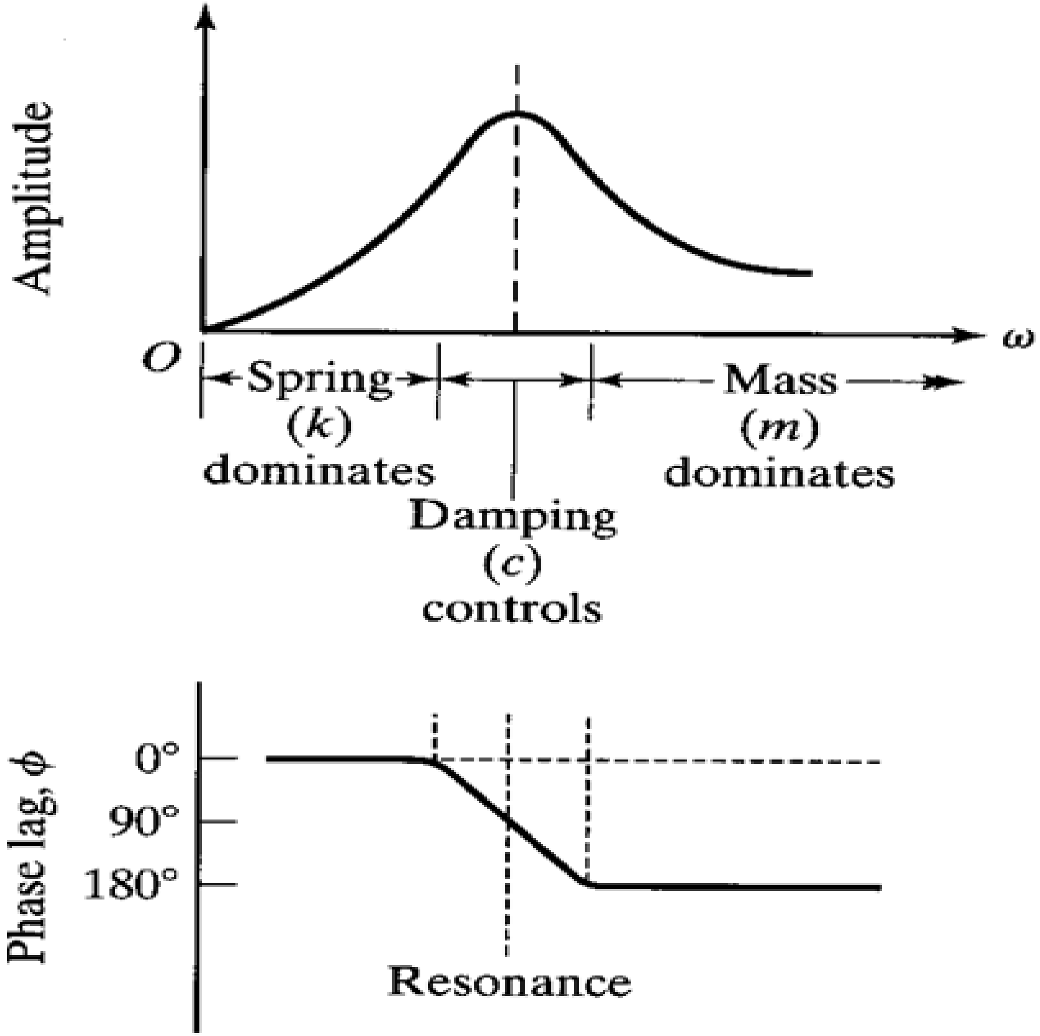
Whirling

$$X = \frac{R(r)^2}{\sqrt{[1 - r^2]^2 + [2\varsigma r]^2}}$$



Critical speed is at $r = 1$ where $\omega = \omega_r = \omega_n = \sqrt{k/m}$

Whirling



Amplitude
O
ω
Spring (k) dominates
Damping (c) controls
Mass (m) dominates
Phase lag, φ
0°
90°
180°
Resonance

Example 3

A rotor having a mass of 5 kg is mounted midway on a 1 cm diameter shaft support at the ends by two bearings. The bearing span is 40 cm. Because of certain manufacturing inaccuracies, the center of gravity of the disc is 0.02 mm away from the geometric center of the rotor. If the system rotates at 3000 rpm, find the amplitude of steady state vibrations and dynamic force transmitted to the bearings. Neglect damping and take $E = 1.96 \times 10^{11}$ N/m^2.

- The moment of inertia of a shaft with diameter $d = 0.01$ m is

$$I = \frac{\pi d^4}{64} = 4.9 \times 10^{-10}\ \text{m}^4;$$

- The simply supported shaft of length $L = 0.4$ m loaded at mid span can be represented by an equivalent spring (where $a = b = L/2$):

$$k_{eq} = \frac{3EI(a+b)}{a^2 b^2} = \frac{48EI}{L^3} = 72.2 \times 10^3\ \text{N/m}$$

Example 3

- Forcing frequency $\omega = 3000 \times 2\pi \div 60 = 314.2$ rad/s;
- Mass $m = 5$ kg and natural frequency $\omega_n = \sqrt{k/m} = 120.2$ rad/s;
- Frequency ratio $r = \omega/\omega_n = 2.615$;
- Given eccentricity $R = 0.02 \times 10^{-3}$ m;
- Amplitude of the steady state vibration with no damping:

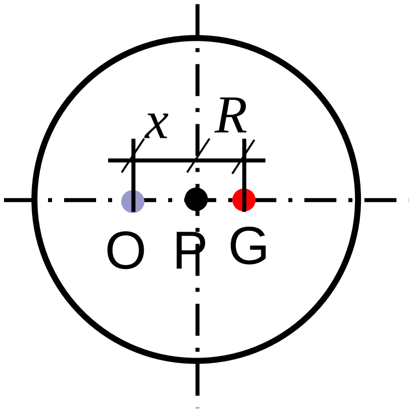
$$X = Y = \frac{R(r)^2}{\sqrt{[1-r^2]^2+[2\varsigma r]^2}} = \frac{R(r)^2}{|1-r^2|} = 0.023 \text{ mm}$$

- The dynamic force transmitted to the bearings

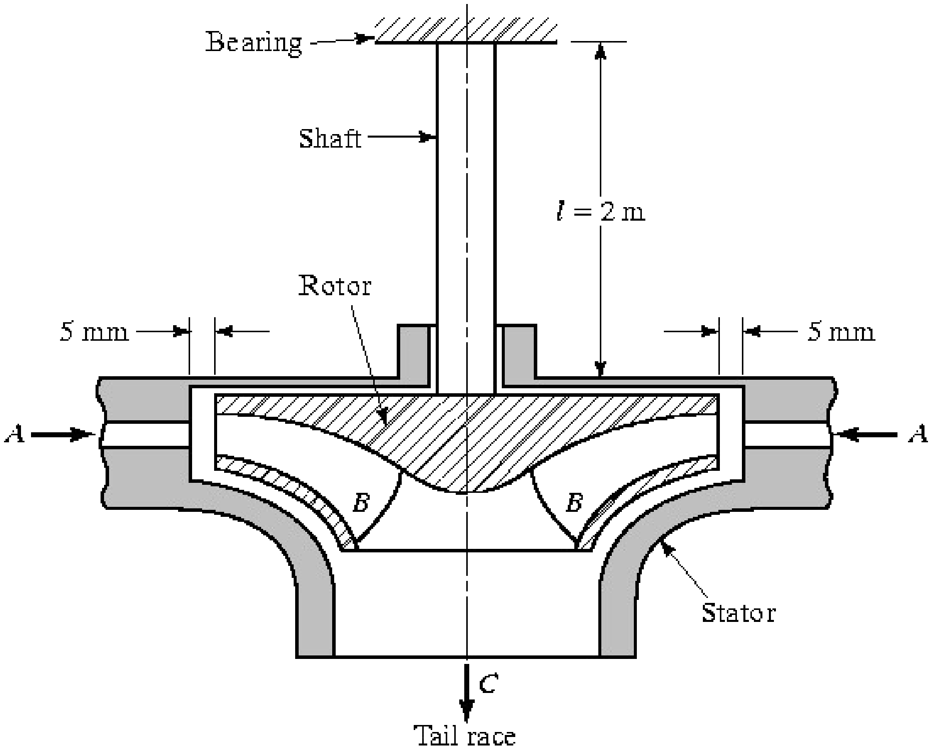
$$F = m(X+R)\omega^2 = 5 \times \left(\frac{0.023 + 0.02}{1000}\right) \times 314.16^2$$

$$F = 21.2 \text{ N};$$

$$F_{\text{for each bearing}} = \frac{21.2}{2} = 10.6 \text{ N}$$



Design example



The schematic diagram of a Francis water turbine is shown. Water flows from *A* into the blades *B* and down into the tail race *C*. The rotor has a mass of 250 kg and an unbalance (m_0e) of 5kg-mm. The radial clearance between the rotor and the stator is 5 mm. The turbine operates in the speed range 600 to 6000 rpm. The steel shaft carrying the rotor can be assumed to be clamped at the bearings. Determine the diameter of the shaft so that the rotor is always clear of the stator at all the operating speeds of the turbine. Assume damping to be negligible. Take $E = 2 \times 10^{11}$ Pa.

Design example

❖ The input frequency ranges from 600 to 6000 rpm or from

$$\omega_1 = 600 \times 2\pi \div 60 = 62.84 \text{ rad/s and to}$$

$$\omega_2 = 6000 \times 2\pi \div 60 = 628.4 \text{ rad/s and}$$

❖ The rotor mass $m = 250$ kg; Natural frequency is

$$\omega_n = \sqrt{k/m} = 0.0632\sqrt{k} \text{ rad/s}$$

❖ The frequency ratio ranges from

$$r_1 = \omega_1/\omega_n = 994/\sqrt{k} \text{ and to}$$

$$r_2 = \omega_2/\omega_n = 9940/\sqrt{k};$$

❖ Negligible damping $\zeta = 0$; Note that the clearance between the rotor and the stator is 5 mm and the steady state amplitude response should not exceed this value

Design example

❖ Steady state response with $\zeta = 0$:

$$X = \frac{r^2\,(m_0 e)/m}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{r^2\,(m_0 e)/m}{1-r^2}$$

❖ Given $m_0 e = 5 \times 10^{-3}$ kg-m;

❖ At $r_1 = 994/\sqrt{k}$: $X = 0.005 = \frac{r_1^2 (m_0 e)/m}{1-r_1^2}$

$$250 = \frac{r_1^2}{1-r_1^2} = \frac{(994)^2}{k-(994)^2}$$

$$250k = (994)^2 + 250(994)^2$$

$$k = 99.2 \times 10^4 \text{ N/m}$$

Design example

- At $r_2 = 9940/\sqrt{k}$: $X = 0.005 = \frac{r_2^2(m_0 e)/m}{1-r_2^2}$

$$250 = \frac{r_2^2}{1-r_2^2} = \frac{(9940)^2}{k-(9940)^2}$$

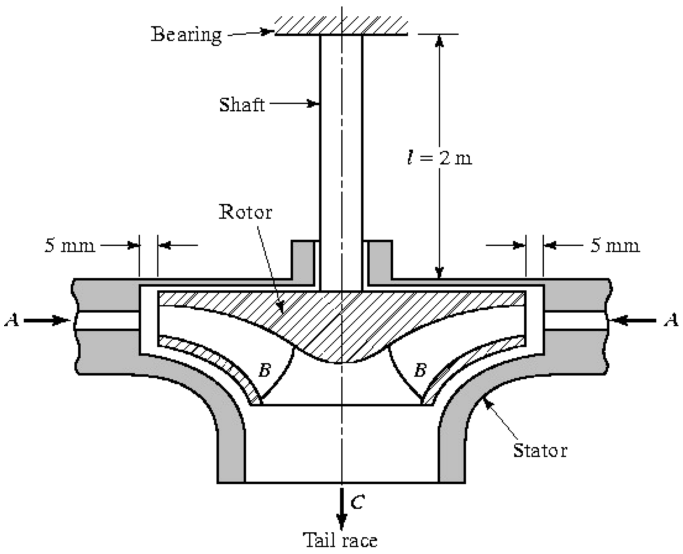
$$250k = (9940)^2 + 250(9940)^2$$

$$k = 99.2 \times 10^6 \text{ N/m}$$

- The rotor and shaft can be modelled as a cantilever beam with end load. The equivalent spring constant is

$$k_{eq} = \frac{3EI}{L^3}$$

$E = 2 \times 10^{11}$ Pa and length $L = 2$ m



Design example

❖ Moment of inertia for a shaft is

$$I = \frac{\pi d^4}{64}\ \text{m}^4;$$

$$k_{eq} = \frac{3EI}{L^3} = \frac{3E}{L^3}\left(\frac{\pi d^4}{64}\right)$$

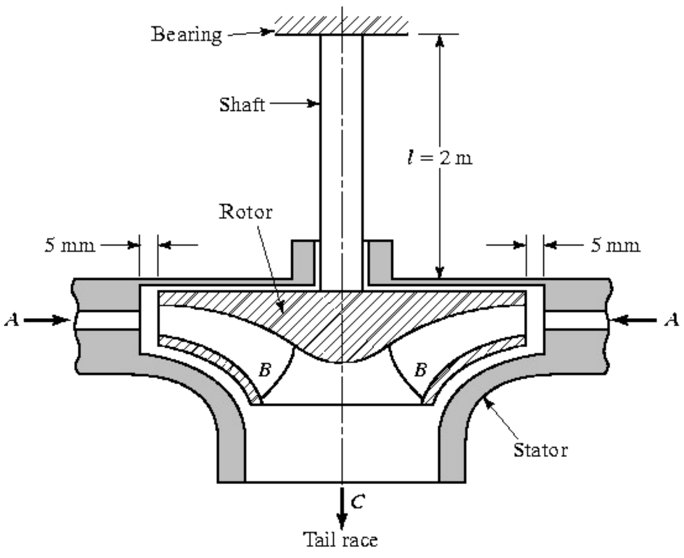
❖ The diameter is given by

$$d^4 = k_{eq}\frac{64L^3}{3\pi E}$$

❖ For $k = 99.2 \times 10^4$ N/m, diameter $d = 128$ mm;

❖ For $k = 99.2 \times 10^6$ N/m, diameter $d = 405$ mm;

Select smaller diameter and check the result



Design example

For $k = 99.2 \times 10^4$ N/m, diameter $d = 128$ mm;

- Natural frequency is $\omega_n = 0.0632\sqrt{k} = 62.95$ rad/s
- Frequency ratio at $\omega_1 = 62.84$ rad/s is $r_1 = 994/\sqrt{k} = 0.998$

$$|X| = \frac{r_1^2 (m_0 e)/m}{1 - r_1^2} = 0.005 \text{ m;}$$

- Frequency ratio at $\omega_2 = 628.4$ rad/s is $r_2 = \omega_2/\omega_n = 9.98$

$$|X| = \frac{r_2^2 (m_0 e)/m}{1 - r_2^2} = 0.00002 \text{ m;}$$

- Therefore the shaft diameter of 128 mm is adequate to limit the steady state vibration amplitude to within 0.005 m. Hence select shaft diameter as 128 mm