ME1020
Mechanical vibrations

Lecture 5

Base excitation

Objectives

- Describe the characteristics of amplitude ratio and phase shift with respect to the frequency ratio
- Analyze the response of 1DOF vibration system to base excitation including displacement transmission and force transmissivity
- Apply the base excitation to seismic instruments using relative motion

Steady state response

A 1DOF system under harmonic excitation: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)$

In the steady state:

$$x(t) = X\cos(\omega t - \theta)$$

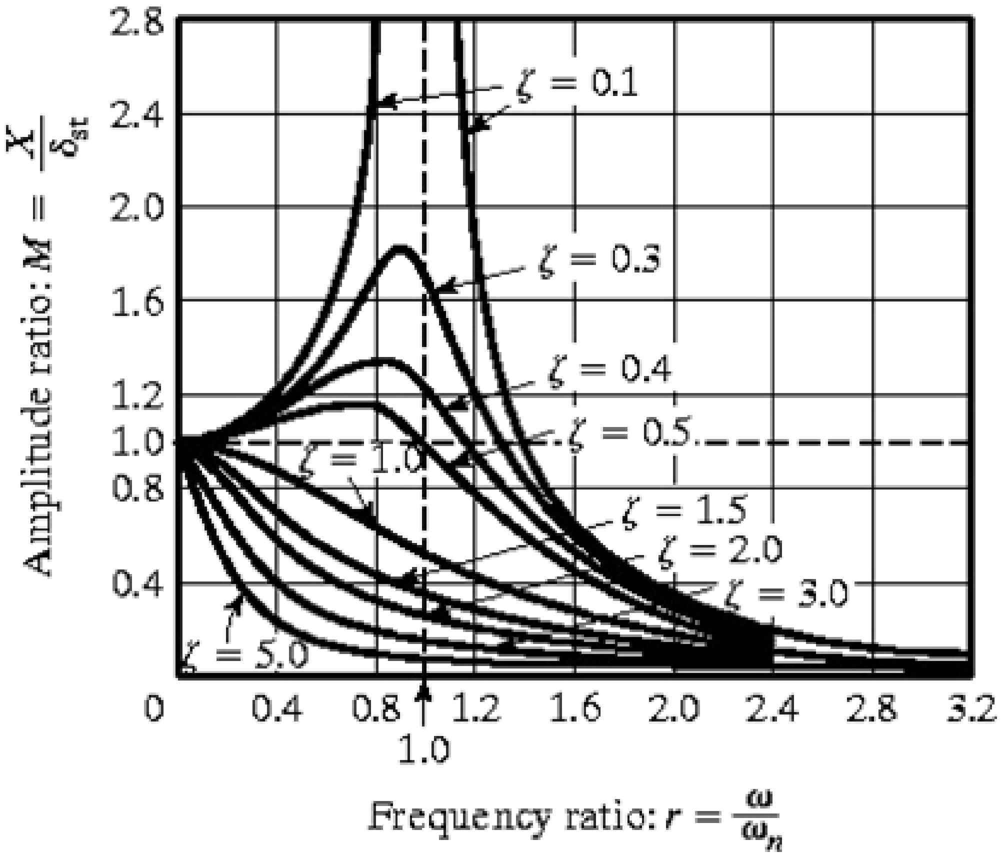
$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}}$$

$$\theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)} = \tan^{-1}\frac{c\omega}{[k - m\omega^2]} = \tan^{-1}\frac{2\zeta r}{[1 - r^2]}$$

Amplitude ratio

$$M = \frac{X}{\delta_{st}} = \frac{kX}{F_0} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} = \frac{1}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}}$$

Amplitude ratio

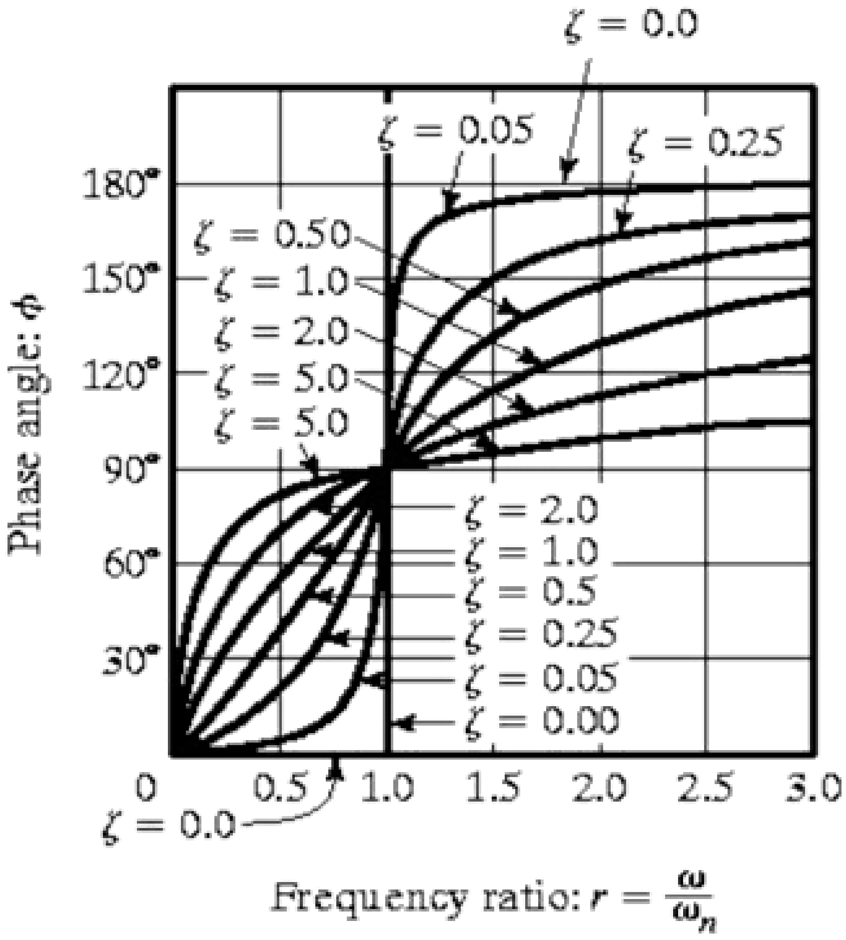


$$M = \frac{1}{\sqrt{[1 - r^2]^2 + [2\zeta r]^2}}$$

Damping has a large influence on the amplitude ratio near resonance where $r = \frac{\omega}{\omega_n} = 1$

Resonance

Phase shift

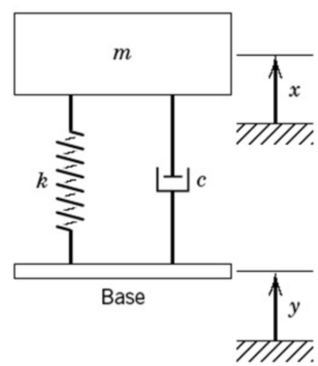


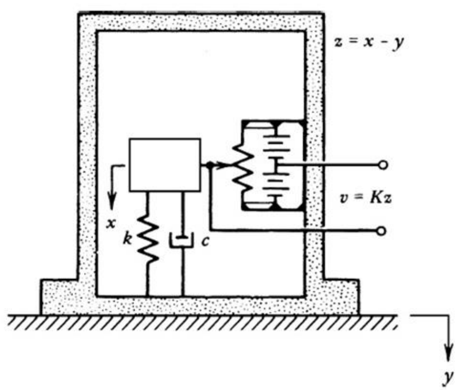
$$\theta = \tan^{-1}\frac{2\zeta r}{[1-r^2]}$$

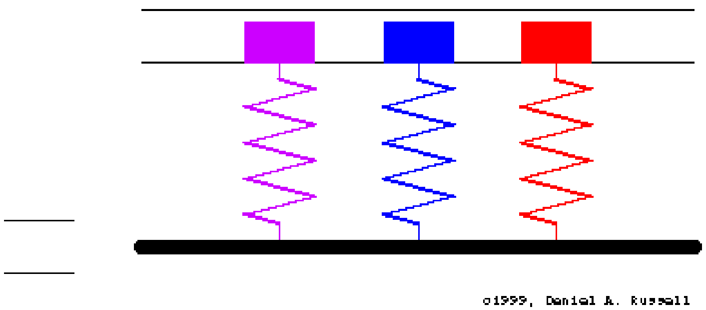
The phase angle is the amount of the cycle by which the motion of the mass lags the forcing function

Base excitation examples

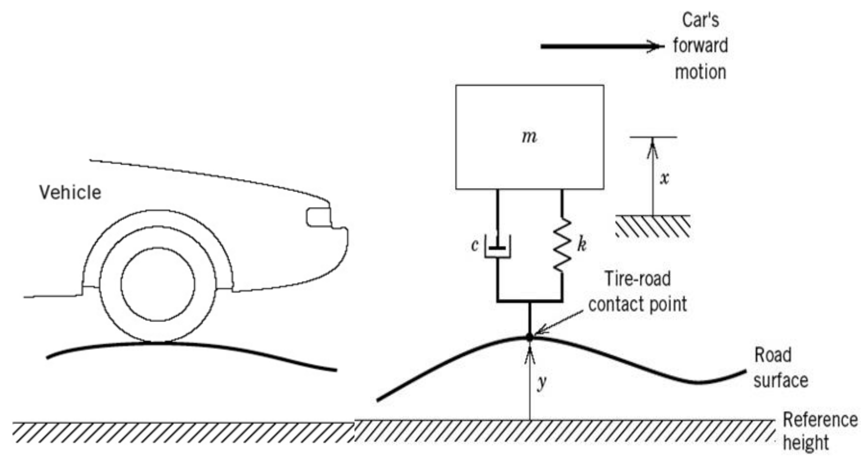
Seismic instruments



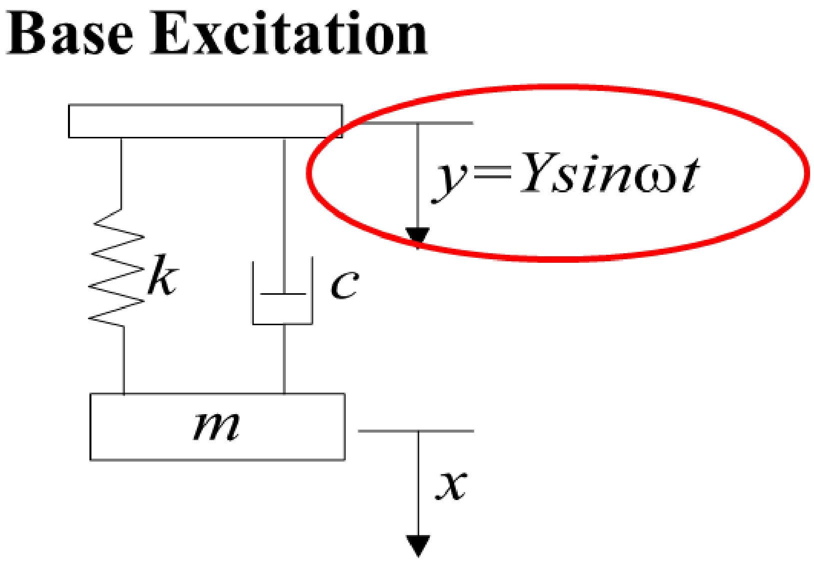


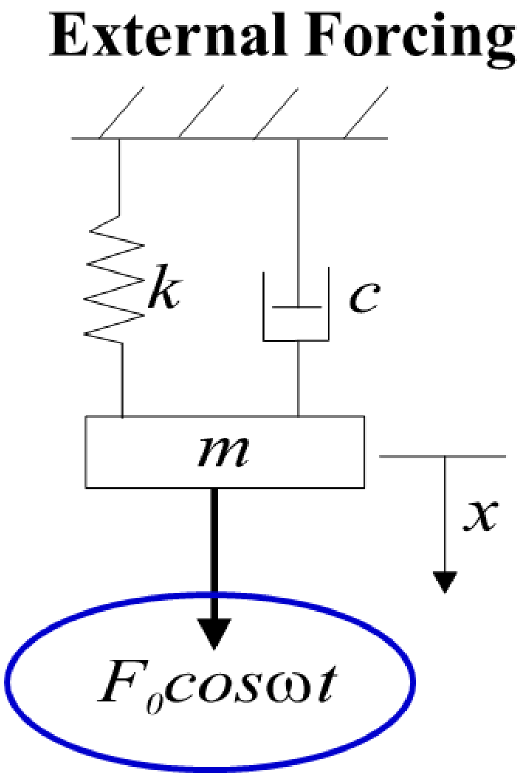


Car suspension

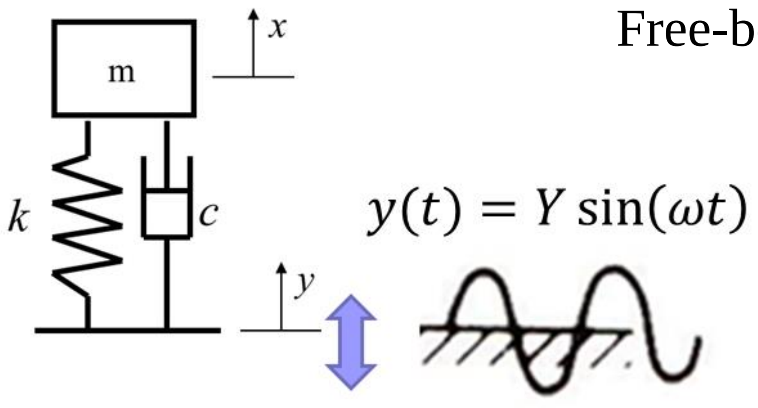


Base vs force excitation

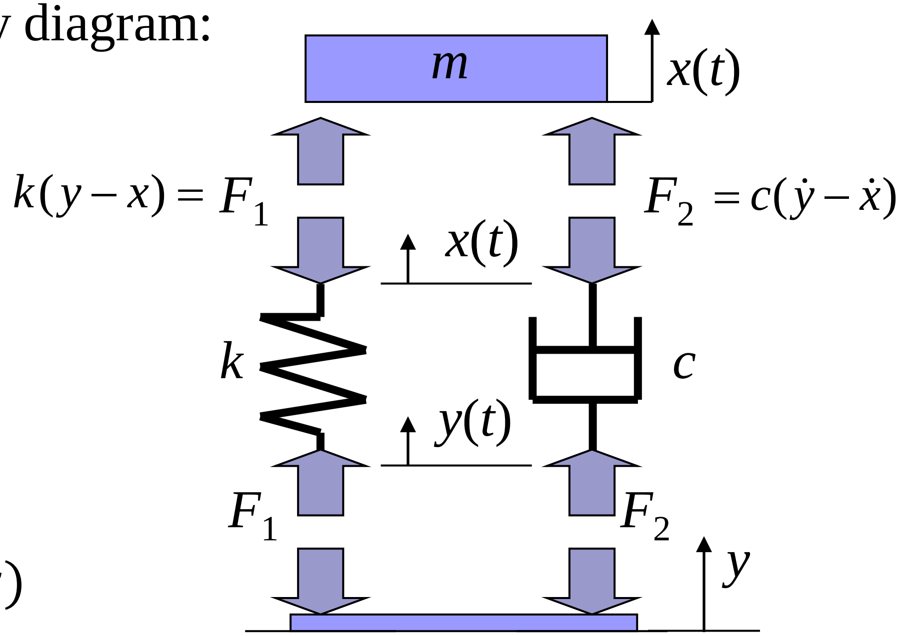




Equation of motion



Free-body diagram:



$$m\ddot{x} = F_1 + F_2$$

$$m\ddot{x} = k(y - x) + c(\dot{y} - \dot{x})$$

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$m\ddot{x} + c\dot{x} + kx = kY\sin(\omega t) + c\omega Y\cos(\omega t)$$

$$m\ddot{x} + c\dot{x} + kx = Y_0\sin(\omega t + \alpha)$$

where $Y_0 = Y\sqrt{k^2 + (c\omega)^2}$ and $\alpha = \tan^{-1}\left[\frac{c\omega}{k}\right]$

Steady state response

For harmonic force excitation: $m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$

- ❖ Steady state response is $x(t) = X \sin(\omega t - \theta)$

where $X = \frac{F_0}{\sqrt{[k-m\omega^2]^2+(c\omega)^2}}$ and $\theta = \tan^{-1}\frac{c\omega}{[k-m\omega^2]}$

For base excitation: $m\ddot{x} + c\dot{x} + kx = Y_0 \sin(\omega t + \alpha)$

- ❖ Steady state response is $x(t) = X \sin(\omega t + \alpha - \theta)$

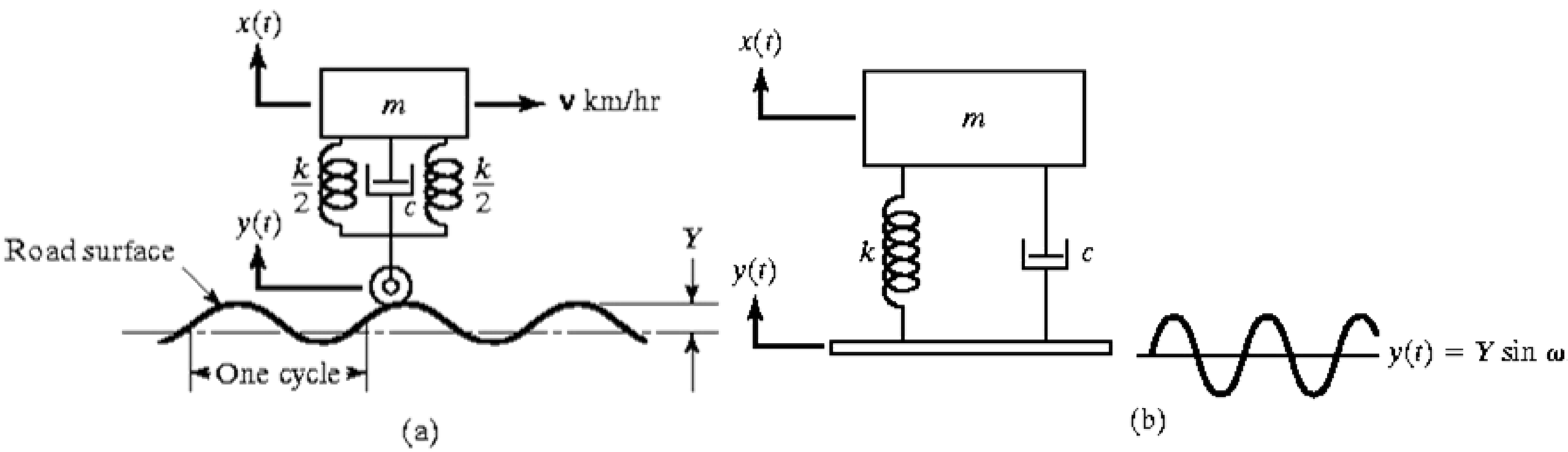
where $X = \frac{Y_0}{\sqrt{[k-m\omega^2]^2+(c\omega)^2}} = \frac{Y\sqrt{k^2+(c\omega)^2}}{\sqrt{[k-m\omega^2]^2+(c\omega)^2}}$ and $\theta = \tan^{-1}\frac{c\omega}{[k-m\omega^2]}$

- ❖ Displacement transmission ratio is defined as

$$\frac{X}{Y} = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

Example 1

The figure below shows a simple model of a motor vehicle that can vibrate in the vertical direction while traveling over a rough road. The vehicle has a mass of 1200 kg. The suspension system has a spring constant of 400 kN/m and a damping ratio of $\zeta = 0.5$. If the vehicle speed is 20 km/hr, determine the displacement amplitude of the vehicle. The road surface varies sinusoidally with an amplitude of $Y = 0.05$m and a wavelength of 6m.



Example 1

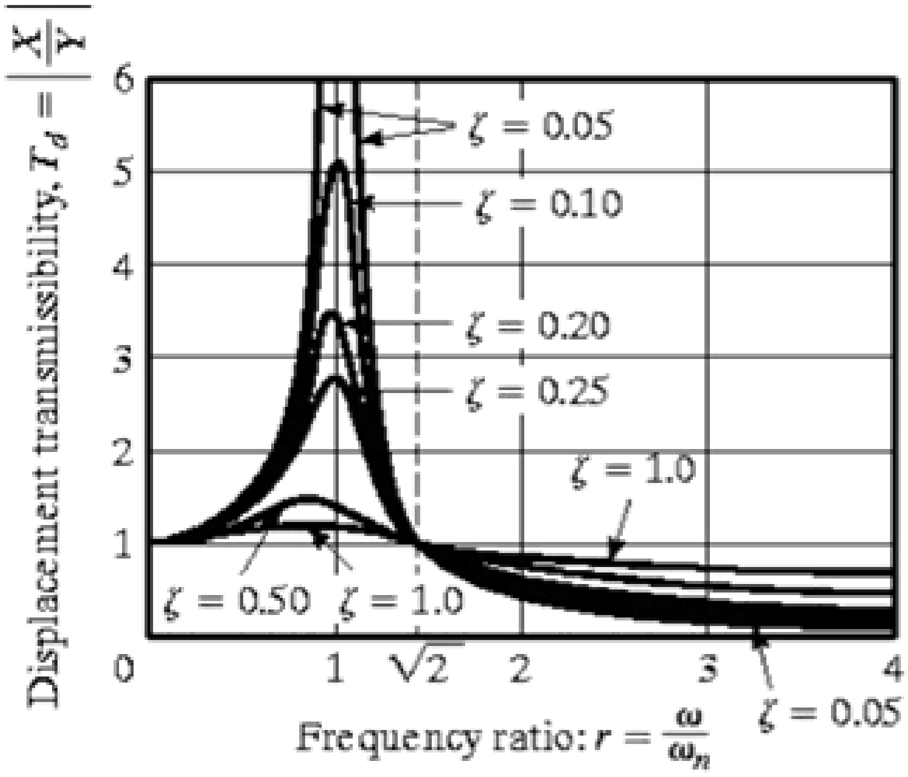
Given $m = 1200$ kg and $k = 400$ kN/m.
Natural frequency is $\omega_n = \sqrt{k/m} = 18.257$ rad/s
Given wavelength $\lambda = v/f = 6$ m, and $v = 20 \times 1000 \div 3600$ m/s
The forcing frequency is $\omega = 2\pi f = 2\pi \frac{v}{\lambda} = 5.82$ rad/s;
Frequency ratio $r = \omega/\omega_n = 0.319$
Damping ratio of $\zeta = 0.5$;
Road amplitude $Y = 0.05$ m
Displacement transmission ratio

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = 1.1 \text{ and } X = 0.055$$

This indicates that a 5cm bump in the road is transmitted as a 5.5 cm bump to the chassis and the passengers of the car

Displacement transmissibility

$$T_d = \frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$



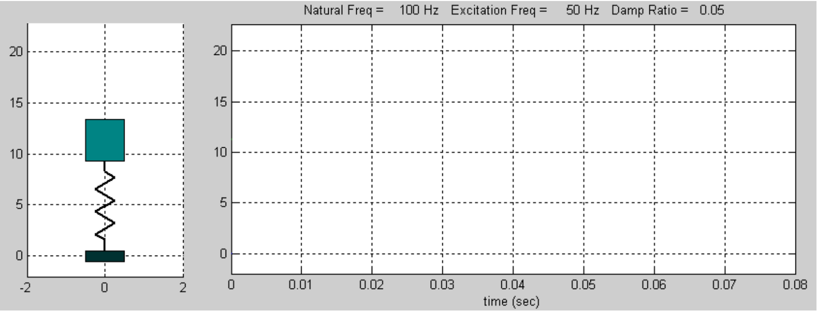
For $0 < \zeta < 1$, T_d attains max at frequency ratio:

$$r_m = \frac{1}{2\zeta}\left[\sqrt{1 + 8\zeta^2} - 1\right]^{1/2}$$

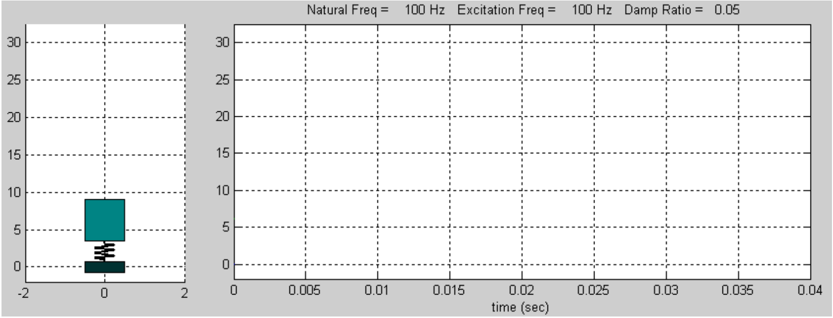
To reduce the displacement transmission, we should have large r or $\omega_n << \omega$

Displacement transmission

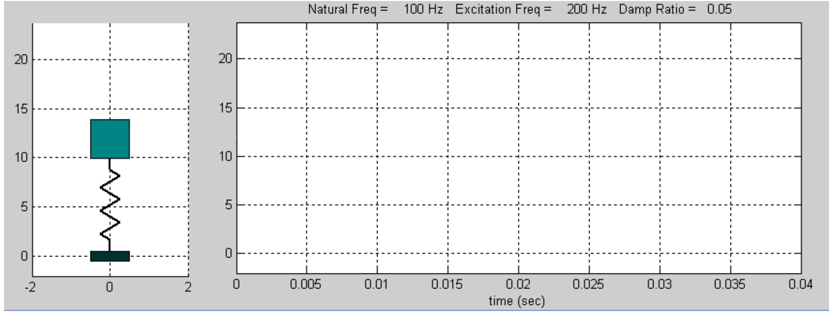
Natural frequency = 100 Hz
Excitation frequency = 50 Hz
Damping ratio = 0.05



Natural frequency = 100 Hz
Excitation frequency = 100 Hz
Damping ratio = 0.05



Natural frequency = 100 Hz
Excitation frequency = 200 Hz
Damping ratio = 0.05



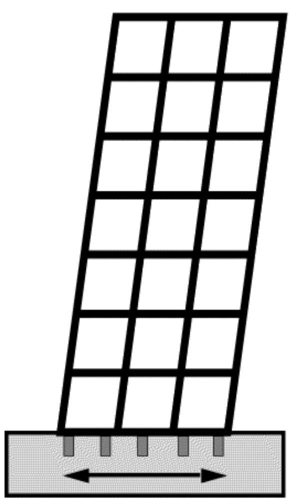
Displacement transmissibility

To reduce the displacement transmission, we should have $\omega_n << \omega$

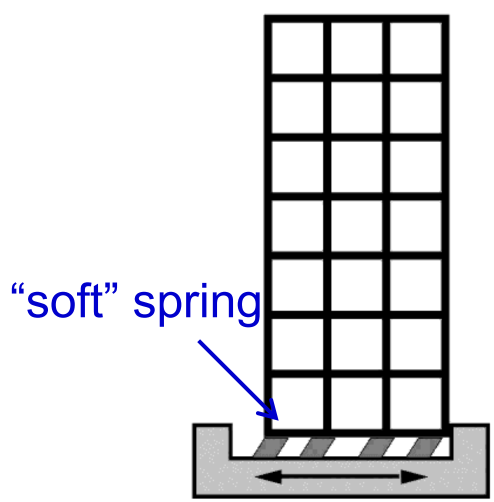
Note that $\omega_n = \sqrt{\frac{k}{m}}$ and small spring constant k (soft spring) leads to small ω_n

- ❖ One way is to isolate the transmission using soft spring

Non-isolated



Isolated



Phase shift

For base excitation: $m\ddot{x} + c\dot{x} + kx = Y_0 \sin(\omega t + \alpha)$

- Steady state response is $x(t) = X \sin(\omega t + \alpha - \theta) = X \sin(\omega t - \phi)$

where $\alpha = \tan^{-1}\left[\frac{c\omega}{k}\right]$ and $\theta = \tan^{-1}\frac{c\omega}{[k - m\omega^2]}$

- Using $\tan(\theta - \alpha) = \frac{\tan(\theta) - \tan(\alpha)}{1 + \tan(\alpha)\tan(\theta)} = \frac{\frac{c\omega}{k - m\omega^2} - \frac{c\omega}{k}}{1 + \frac{c\omega}{k}\left(\frac{c\omega}{k - m\omega^2}\right)} = \frac{\frac{c\omega k - c\omega(k - m\omega^2)}{k(k - m\omega^2)}}{1 + \frac{(c\omega)^2}{k(k - m\omega^2)}}$

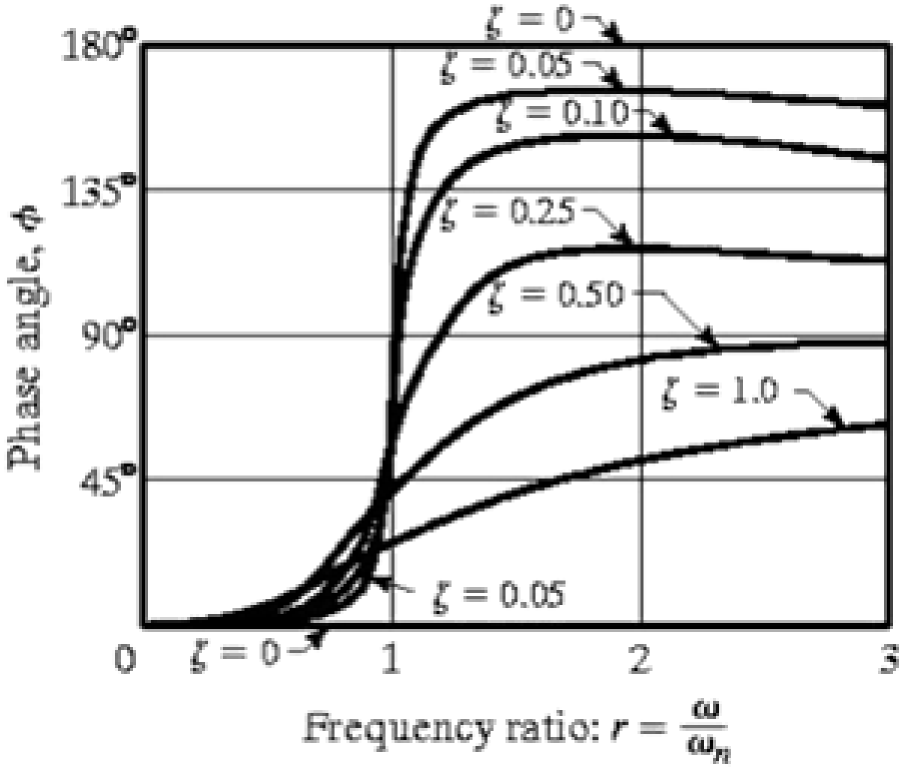
$$\tan(\theta - \alpha) = \frac{c\omega k - c\omega(k - m\omega^2)}{k(k - m\omega^2) + (c\omega)^2} = \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2}$$

- Let $\phi = \theta - \alpha$

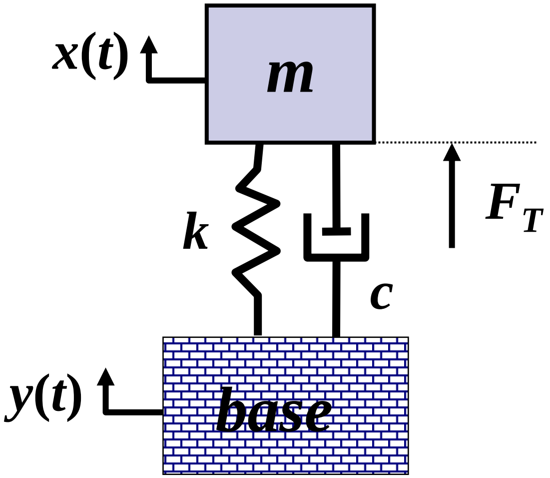
$$\phi = \tan^{-1}\left[\frac{mc\omega^3}{k(k - m\omega^2) + (\omega c)^2}\right] = \tan^{-1}\left[\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right]$$

Phase shift

$$\phi = \tan^{-1}\left[\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right]$$



Force transmitted



- Force is transmitted from the moving base to the mass through the spring and damper
- The transmitted force must balance the inertia force of the mass

$$m\ddot{x} = k(y - x) + c(\dot{y} - \dot{x}) = -F$$

- In the steady state

$$x(t) = X\sin(\omega t - \theta)$$
$$\dot{x}(t) = \omega X\cos(\omega t - \theta)$$
$$\ddot{x}(t) = -\omega^2 X\sin(\omega t - \theta)$$

Therefore $F = -m\ddot{x} = m\omega^2 X\sin(\omega t - \theta) = F_T\sin(\omega t - \theta)$

F_T is the magnitude of the transmitted force

$$F_T = m\omega^2 X = k\left(\frac{\omega}{\omega_n}\right)^2 X = kr^2 X$$

Force transmitted

- F_T can be simplified using the displacement transmission

$$T_d = \frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

- Substituting this into F_T

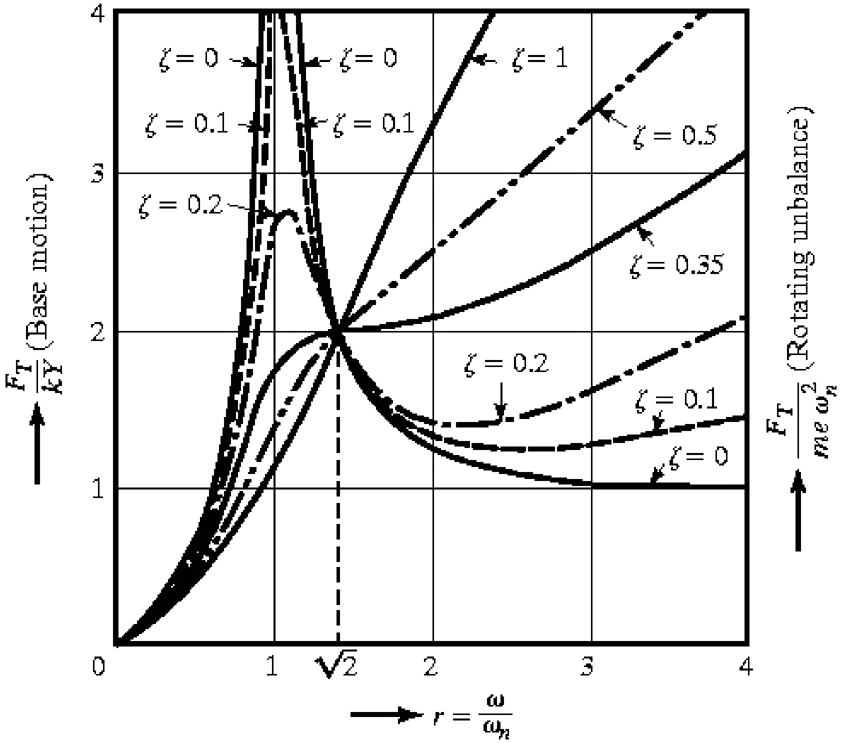
$$F_T = kr^2 X = kr^2 Y \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

- The force transmission ratio is defined as

$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

Force transmissibility

$$\frac{F_T}{kY} = r^2 \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$



Example 2

Given a base-excitation system with mass $m = 100$ kg, damping coefficient $c = 30$ kg/s, and spring constant $k = 200$ N/mm, magnitude of base displacement $Y = 0.03$ m, and forcing frequency $\omega = 6$ rad/s. Compute the magnitude of the displacement transmissibility ratio and the force transmissibility ratio.

Natural frequency is $\omega_n = \sqrt{k/m} = 4.472$ rad/s

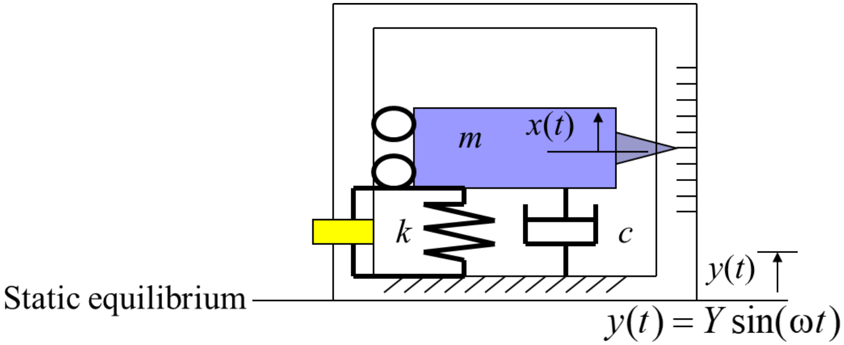
Damping ratio $\zeta = \frac{c}{2\sqrt{mk}} = 0.034$,

Frequency ratio $r = \omega/\omega_n = 1.342$

Displacement transmissibility ratio $T_d = \frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = 0.557$

Force transmissibility ratio $\frac{F_T}{kY} = r^2\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = 1.003$

Relative motion



- Relative motion is important in seismic instrument
- For example, the displacement transducer measures $z(t) = x(t) - y(t)$ to determine the base acceleration
- The equation of motion for the transducer can be written as

$$m\ddot{x} = c(\dot{y} - \dot{x}) + k(y - x)$$

- Introducing the relative displacement $z(t) = x(t) - y(t)$:

$$m\ddot{x} - m\ddot{y} = c(\dot{y} - \dot{x}) + k(y - x) - m\ddot{y}$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

- For $y(t) = Y\sin(\omega t) \Longrightarrow \ddot{y}\,(t) = -\omega^2 Y \sin(\omega t)$ and

$$m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin(\omega t)$$

Relative motion

For harmonic force excitation: $m\ddot{x} + c\dot{x} + kx = F_0 \sin(\omega t)$

❖ Steady state response is $x(t) = X \sin(\omega t - \theta)$

where $X = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}}$ and $\theta = \tan^{-1}\frac{c\omega}{[k - m\omega^2]}$

For relative motion: $m\ddot{z} + c\dot{z} + kz = m\omega^2 Y \sin(\omega t)$

❖ Steady state response is $z(t) = Z \sin(\omega t - \theta)$

where $Z = \frac{m\omega^2 Y}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}} = \frac{Yr^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$

and $\theta = \tan^{-1}\frac{c\omega}{[k - m\omega^2]} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$

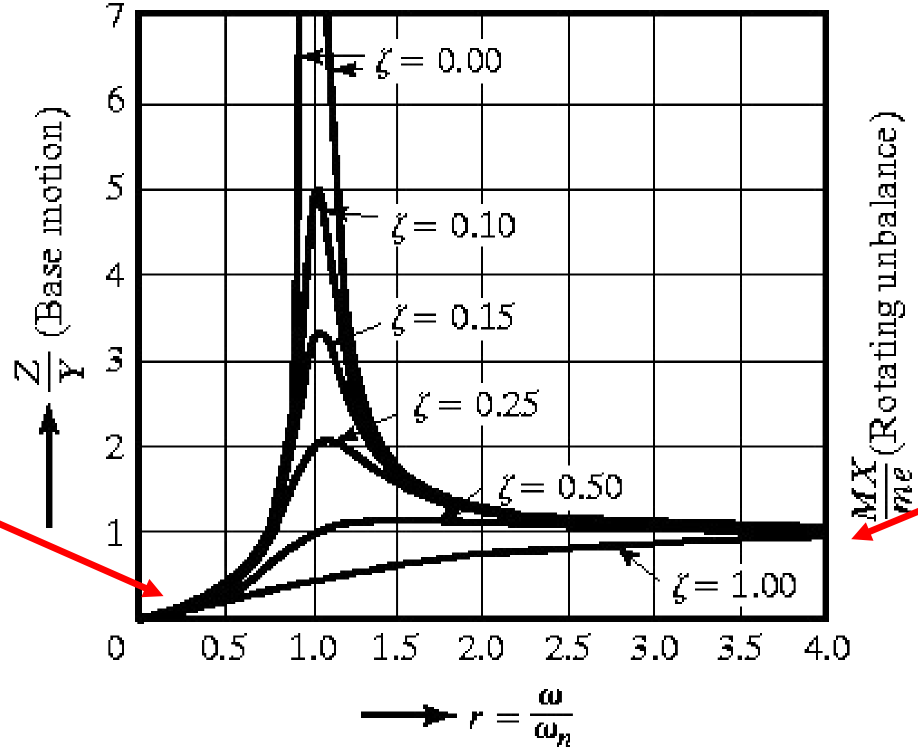
❖ Amplitude ratio for relative motion is defined as

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Relative motion

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Low frequencies: relative motion approaches zero



High frequencies: relative motion approaches 1

Accelerometer

- Accelerometer measures the acceleration of the vibrating body
- Actual ground motion is $y(t) = Y\sin(\omega t)$
- Actual ground acceleration is $\ddot{y}(t) = -\omega^2 Y\sin(\omega t)$

$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{\omega^2}{\omega_n^2}\frac{Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\omega^2 Y = \omega_n^2 Z\sqrt{(1-r^2)^2 + (2\zeta r)^2}$$

- Maximum actual acceleration amplitude is $\omega^2 Y$
- Maximum measured acceleration amplitude is $\omega_n^2 Z$
- The absolute measured value to the absolute true value is

$$\frac{\text{absolute measured value}}{\text{absolute true value}} = \frac{\omega_n^2 Z}{\omega^2 Y} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Accelerometer

$$\frac{\text{absolute measured value}}{\text{absolute true value}} = \frac{\omega_n^2 Z}{\omega^2 Y} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

- When measured value ≈ true value:

$$\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \approx 1$$

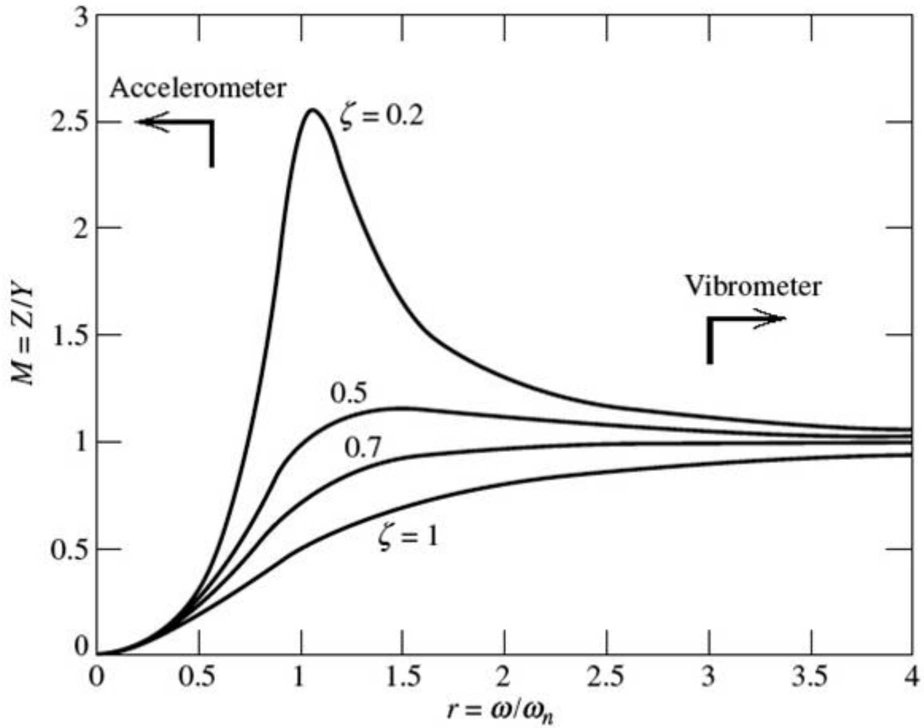
- Error between measured and true values is defined as

$$e = \frac{\omega_n^2 Z - \omega^2 Y}{\omega^2 Y} = \frac{\omega_n^2 Z}{\omega^2 Y} - 1$$

$$e = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} - 1$$

Vibrometer

- Vibrometer measures the displacement of a vibrating body
- $Z/Y \approx 1$ for $r = \omega/\omega_n > 3$ so that $Z \approx Y$



Example 3

An accelerometer has a suspended mass of 0.01 kg with a damped natural frequency of vibration of 150 Hz. When mounted on an engine undergoing an acceleration of 1g at an operating speed of 6000 rpm, the acceleration is recorded as 9.5 m/s^2 by the instrument. Find the damping constant and the spring stiffness of the accelerometer

- Actual acceleration of 1 g is 9.81 m/s^2;
- Measured acceleration is 9.5 m/s^2;

$$\frac{\text{absolute measured value}}{\text{absolute true value}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{9.5}{9.81} = 0.9684$$

$$(1-r^2)^2 + (2\zeta r)^2 = 1.0663$$

- Given operating speed of 6000 rpm and $\omega = 6000\frac{2\pi}{60} = 628.32$ rad/s
- Given damped natural frequency 150 Hz' therefore

$$\omega_d = 2\pi(150) = 942.48 = \omega_n\sqrt{1-\zeta^2}$$

Example 3

- $\omega = 628.32$ rad/s and $\omega_n\sqrt{1-\zeta^2} = 942.48$

$$\frac{\omega}{\omega_n\sqrt{1-\zeta^2}} = \frac{r}{\sqrt{1-\zeta^2}} = \frac{628.32}{942.48} = 0.6667$$

$$r = 0.6667\sqrt{1-\zeta^2} \text{ or } r^2 = 0.4441(1-\zeta^2)$$

- But $(1-r^2)^2 + (2\zeta r)^2 = 1.0663$
- Combine the 2 equations: $1.5801\zeta^4 - 2.2714\zeta^2 + 0.7576 = 0$
- Solve the quadratic equation to get $\zeta^2 = 0.526$ and $\zeta^2 = 0.9115$
- Or $\zeta = 0.7253$ and 0.9547 (2 possible solution)
- For $\zeta = 0.7253$, $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 1368.8889$ rad/s; given $m = 0.01$ kg;
- Spring constant $k = m\omega_n^2 = 18738.5628$ N/m
- Damping coefficient $c = 2\zeta\omega_n m = 19.5871$ Ns/m;