ME1020
Mechanical vibrations

Lecture 4

Response to harmonic excitation (1DOF system)

Objectives

- Derive the 1DOF vibration system model subjected to harmonic excitation using Lagrange’s equation
- Analyze the response of 1DOF undamped vibration system to harmonic excitation
- Analyze the response of 1DOF damped vibration system to harmonic excitation
- Determine the steady state response of 1DOF system in terms of amplitude ratio and phase shift

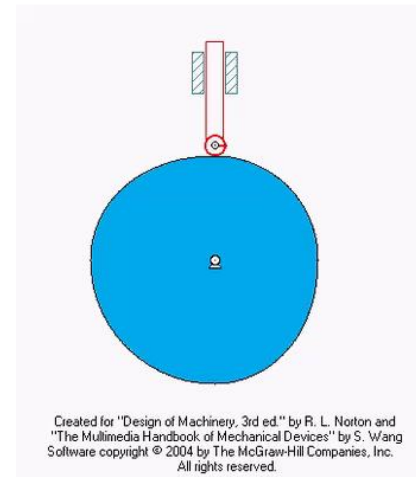
Introduction

- ❖ In free vibrations, initial conditions were used to excite the system
- ❖ In forced vibrations, external energy is applied to “excite” the system
- ❖ The external excitation can be supplied through an applied oscillating force, which may be harmonic, non harmonic but periodic, non periodic, or random in nature.
- ❖ The external excitation encountered in engineering systems is commonly produced by:
  - ▪ Rotating machines, reciprocating machines, etc.
  - ▪ Excitation by another vibrating system
  - ▪ Excitation by natural forces (i.e. earthquake, vortex shedding)
- ❖ Harmonic response results when the system responses to a harmonic excitation

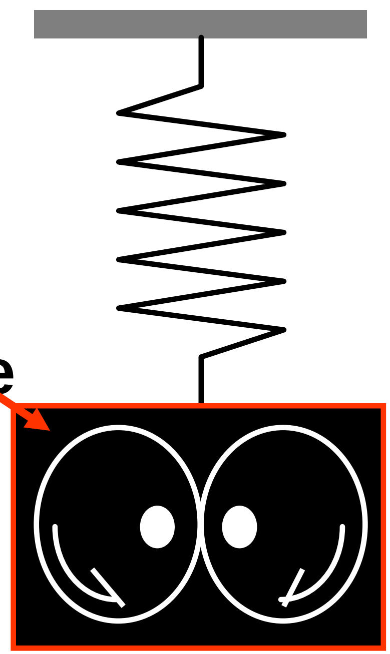
Introduction

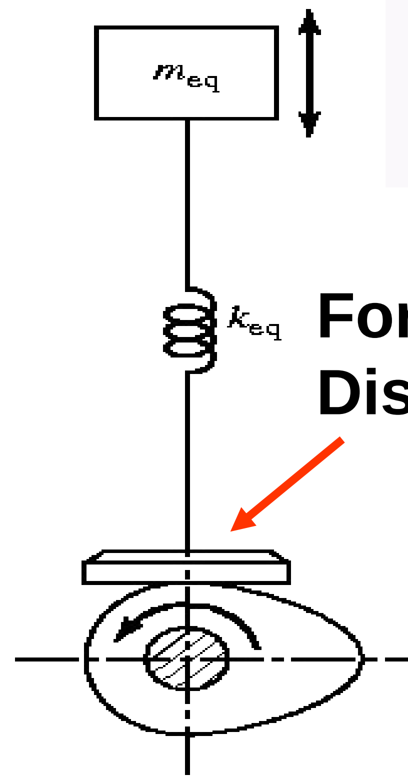
Harmonic excitations can be:

- ❖ Harmonic force
- ❖ Harmonic displacement



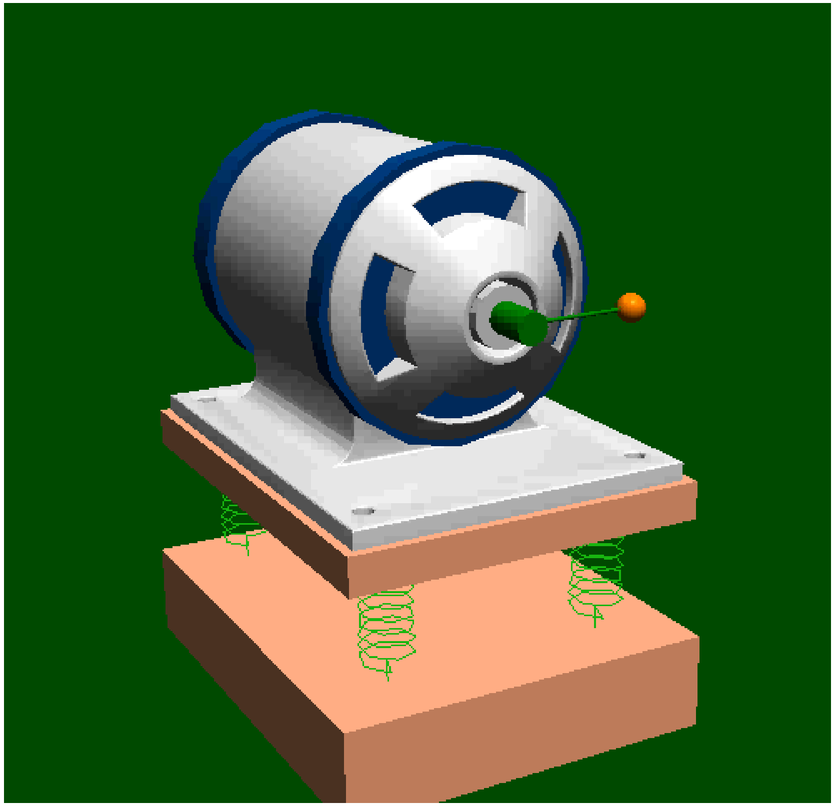
Forcing Force





Forcing Displacement

Harmonic force



Harmonic force excitation can be modeled with a sine or cosine function:

$$F = F_0 \cos(\omega t)$$

where

- ω is the forcing function frequency in rad/s
- F_0 is the forcing function amplitude in N

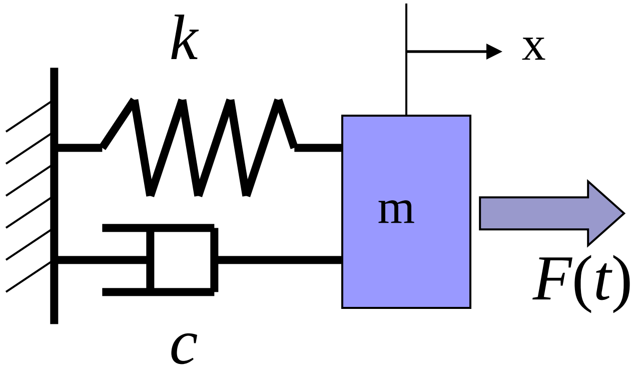
Lagrange's equation

In terms of generalized coordinate q, the Lagrange's equation for a single DOF system subject to a generalized force has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = Q$$

- T = Kinetic energy
- U = Potential energy
- D = Rayleigh's damping (or dissipation) function
- q = generalized coordinate that completely describe the dynamical system
- $Q = \sum_l F_l \cdot \frac{\partial r_l}{\partial q} + \sum_l M_l \cdot \frac{\partial \omega_l}{\partial \dot{q}}$ (note dot product) generalized force for the l bodies
- F_l and M_l are the vector representation of the external applied forces and moments on the lth body, respectively; r_l and ω_l are the position and angular velocity vectors due to F_l and M_l respectively

Lagrange's equation



Generalized coordinate $q = x$

Kinetic energy: $T = \frac{1}{2}m\dot{x}^2$

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial T}{\partial q} = \frac{\partial T}{\partial x} = 0$$

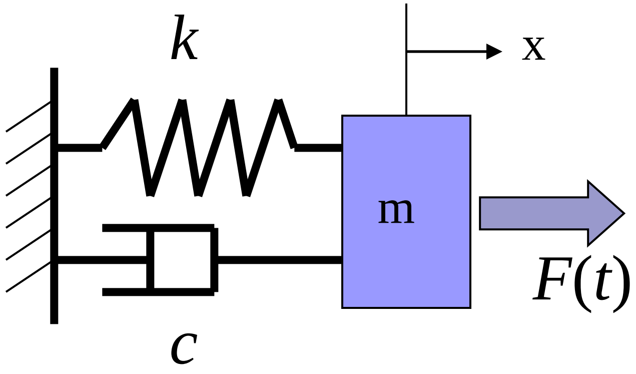
- Dissipation function $D = \frac{1}{2}c\dot{x}^2$

$$\frac{\partial D}{\partial \dot{q}} = \frac{\partial D}{\partial \dot{x}} = c\dot{x}$$

- Potential energy $U = \frac{1}{2}kx^2$

$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial \theta} = kx$$

Lagrange's equation



Generalized force

$$Q = F\frac{\partial r_1}{\partial q} = F(t)\frac{\partial x}{\partial x} = F(t)$$

❖ Apply Lagrange's equation:

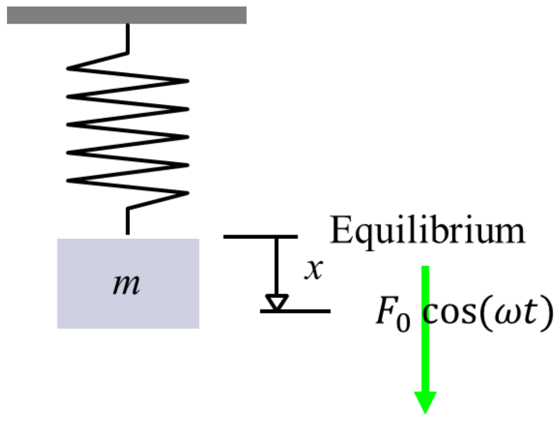
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = Q$$

$$\frac{d}{dt}(m\dot{x}) + c\dot{x} + kx = F(t)$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

Harmonic excitation: undamped

Consider the usual spring-mass system with applied force $F(t) = F_0 \cos(\omega t)$:



Undamped system under harmonic excitation:

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

The equation can also be expressed as:

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)$$

- ❖ Natural frequency $\omega_n = \sqrt{k/m}$
- ❖ Normalized force amplitude $f_0 = \frac{F_0}{m}$

Harmonic excitation: undamped

The undamped system subject to harmonic excitation:

$$\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)$$

- ❖ The solution of the equation has two parts:

$$x(t) = (\text{complementary solution}) + \{\text{particular solution}\}$$

- A complementary solution for the homogenous equation for $F(t) = 0$
- A particular solution irrespective of the free damped-vibration for $F(t) = F_0 \cos(\omega t)$

- ❖ For $F(t) = 0 \Longrightarrow \ddot{x} + \omega_n^2 x = 0$
- ▪ Complementary solution has the form

$$x_C(t) = A \cos(\omega_n t + \phi)$$

- ▪ Which can be rewritten as $x_C(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t)$
- ▪ A_1 and A_2 will be determined from the initial conditions x_0 and v_0 later

Harmonic excitation: undamped

- For $F(t) = F_0 \cos(\omega t) \Longrightarrow \ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)$
- The particular solution is assumed to be related to the forcing function:

$$x_P(t) = X \cos(\omega t)$$

  - Frequency will be the same with forcing function frequency
  - Magnitude of response may be different from the forcing function and X is the amplitude of the steady state response

$$\dot{x}_P(t) = -\omega X \sin(\omega t) \quad \text{and} \quad \ddot{x}_P(t) = -\omega^2 X \cos(\omega t)$$

Substituting these into the differential equation $\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)$:

$$-\omega^2 X \cos(\omega t) + \omega_n^2 X \cos(\omega t) = f_0 \cos(\omega t)$$

$$-\omega^2 X + \omega_n^2 X = f_0$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2}$$

- Particular solution is $x_P(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t)$

Harmonic excitation: undamped

$$x(t) = x_C(t) + x_p(t) = A_1 \sin(\omega_n t) + A_2 \cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t)$$

$$\dot{x}(t) = \omega_n A_1 \cos(\omega_n t) - \omega_n A_2 \sin(\omega_n t) - \frac{\omega f_0}{\omega_n^2 - \omega^2}\sin(\omega t)$$

- At time $t = 0$, $x(0) = x_0$

$$x_0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2}$$

$$A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2}$$

- At time $t = 0$, $\dot{x}(0) = v_0$

$$v_0 = \omega_n A_1 \quad \text{or} \quad A_1 = \frac{v_0}{\omega_n}$$

❖ The total solution is

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t)$$

Harmonic excitation: undamped

For an undamped 1DOF system subject to harmonic excitation:

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

This can be written as

$$\ddot{x} + \omega_n^2 x = f_0 \cos(\omega t)$$

The total solution is:

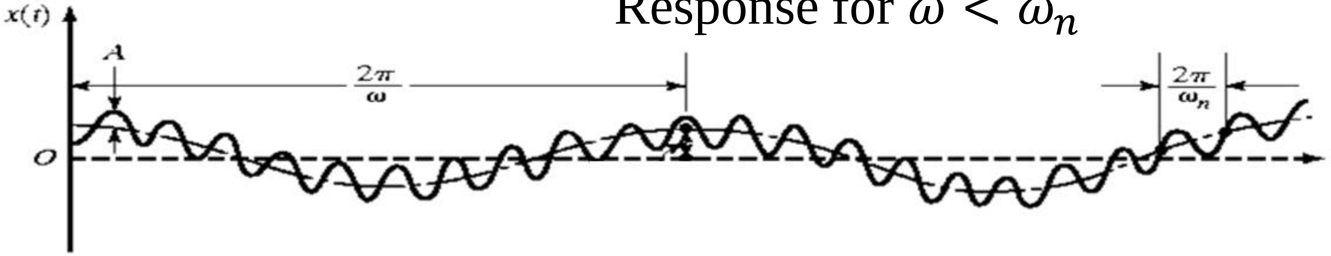
$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos(\omega_n t) + \frac{f_0}{\omega_n^2 - \omega^2}\cos(\omega t)$$

or can also be written as

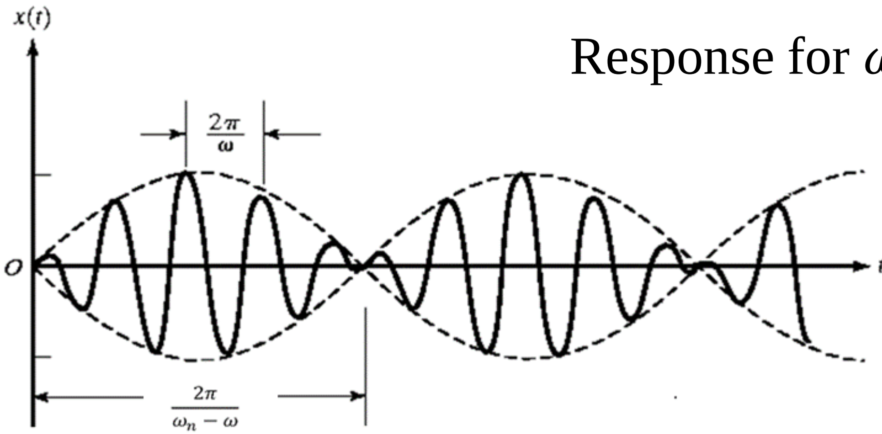
$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos(\omega_n t) + \frac{F_0}{k - m\omega^2}\cos(\omega t)$$

Note: harmonic terms with 2 different frequencies but not defined for $\omega = \omega_n$

Response for $\omega < \omega_n$

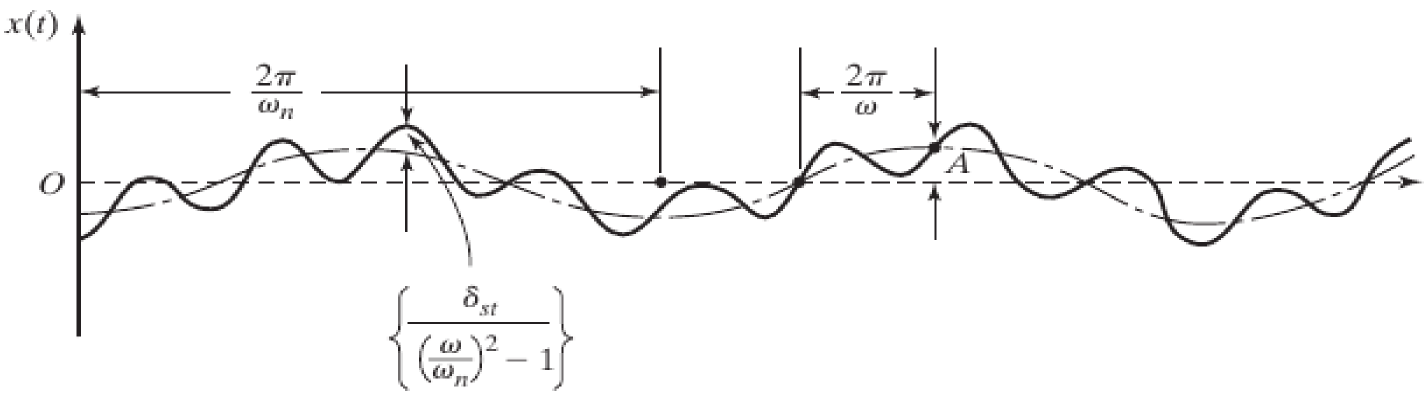


Response for ω approaches ω_n

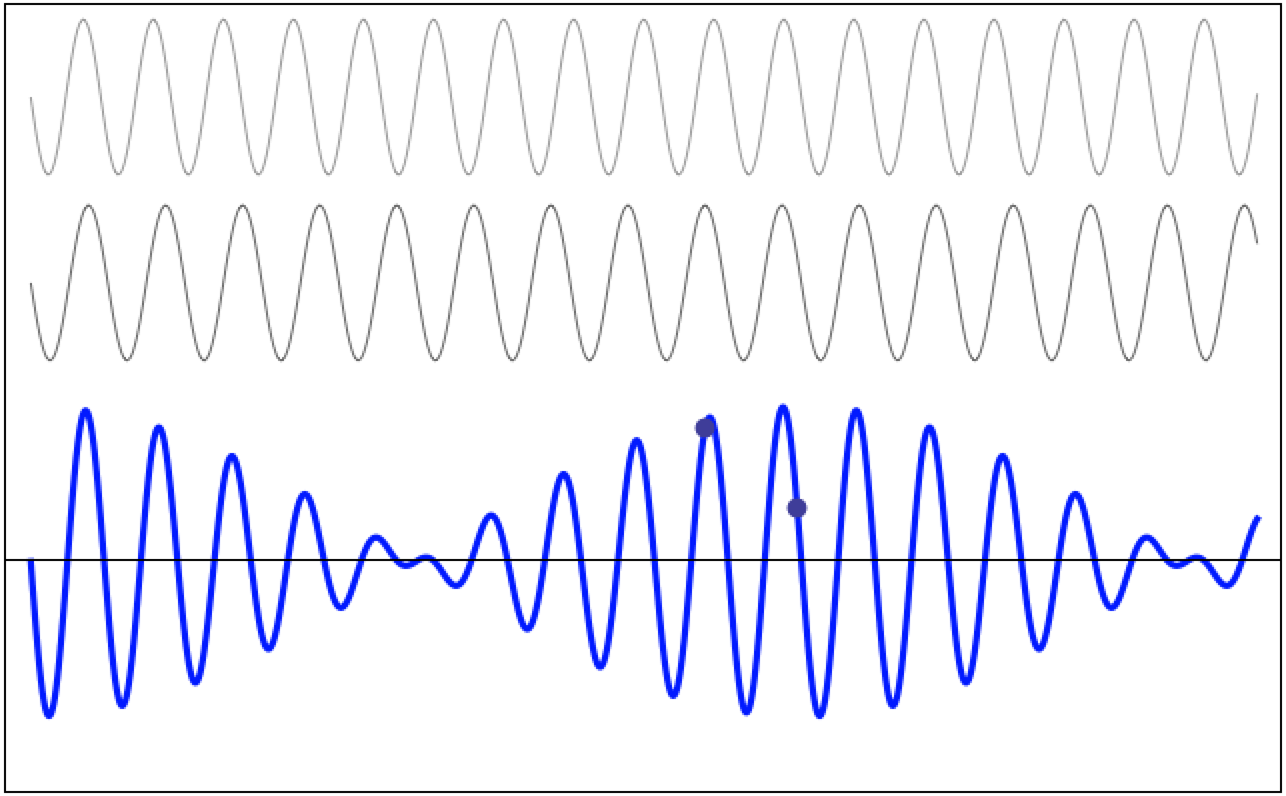


- If ω close to ω_n, then beating occurs
- Beat frequency is $\omega_b = \omega_n - \omega$

Response for $\omega > \omega_n$



Beats



Harmonic excitation: undamped

- Resonance occurs when $\omega = \omega_n$ and the magnitude $x(t)$ becomes infinite
- At this condition, the solution

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos(\omega_n t) + \frac{F_0}{k - m\omega^2}\cos(\omega t)$$

Can be rewritten as

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) + \left(\frac{F_0}{k - m\omega^2}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) + \left(\frac{(F_0/k)}{1 - (m\omega^2/k)}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

Define static deflection $\delta_{st} = (F_0/k)$ and note: $\omega_n^2 = (k/m)$

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) + \left(\frac{\delta_{st}}{1 - (\omega^2/\omega_n^2)}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

Harmonic excitation: undamped

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) + \left(\frac{\delta_{st}}{1-(\omega^2/\omega_n^2)}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

$$\lim_{\omega\to\omega_n}\left\{\frac{\cos(\omega t) - \cos(\omega_n t)}{1-(\omega^2/\omega_n^2)}\right\} = \lim_{\omega\to\omega_n}\left\{\frac{\frac{d}{d\omega}(\cos(\omega t) - \cos(\omega_n t))}{\frac{d}{d\omega}\left(1-(\omega^2/\omega_n^2)\right)}\right\}$$

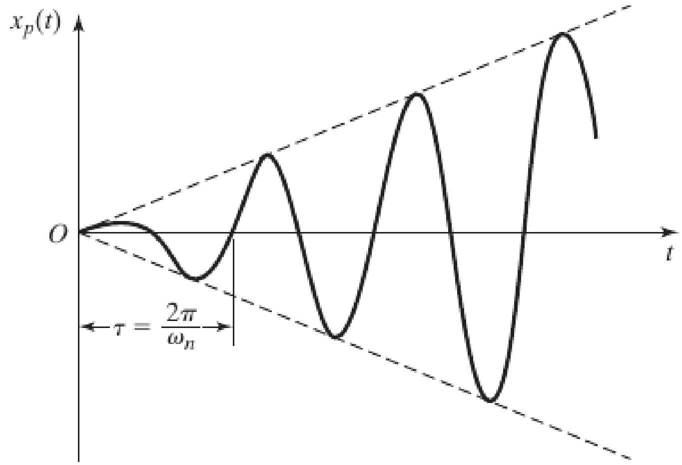
$$= \lim_{\omega\to\omega_n}\left\{\frac{t\sin\omega t}{(2\omega/\omega_n^2)}\right\} = \frac{\omega_n t}{2}\sin\omega_n t$$

The response at resonance when $\omega = \omega_n$ is

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + x_0\cos(\omega_n t) + \delta_{st}\frac{\omega_n t}{2}\sin\omega_n t$$

When $\omega = \omega_n$,

$\delta_{st} \frac{\omega_n t}{2} \sin \omega_n t$ will increases indefinitely



$$k_{eq} = \frac{3EI}{2L^3}$$

$$\omega_n = \sqrt{\frac{3EI}{2mL^3}}$$

Example 1

Given zero initial conditions a harmonic input of 10 Hz with 20 N magnitude and k = 2000 N/m, and measured response amplitude of 0.1m, compute the mass of the system.

Note: input frequency $f = 10$ Hz or $\omega = 2\pi f = 20\pi$ rad/s; and $F_0 = 20$ N;

For a spring-mass system subjected to a harmonic input, the total solution is:

$$x(t) = \frac{v_0}{\omega_n}\sin(\omega_n t) + \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos(\omega_n t) + \frac{F_0}{k - m\omega^2}\cos(\omega t)$$

Note that for initial conditions x_0 and v_0 equalled to zero;

$$x(t) = \left(\frac{F_0}{k - m\omega^2}\right)(\cos(\omega t) - \cos(\omega_n t))$$

$$= \left(\frac{\delta_{st}}{1 - (\omega^2/\omega_n^2)}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

Example 1

$$x(t) = \left(\frac{\delta_{st}}{1 - (\omega^2/\omega_n^2)}\right)\{\cos(\omega t) - \cos(\omega_n t)\}$$

Static deflection $\delta_{st} = (F_0/k) = (20/2000) = 0.01$ m;

Given spring constant $k = 2000\ \mathrm{N/m}$

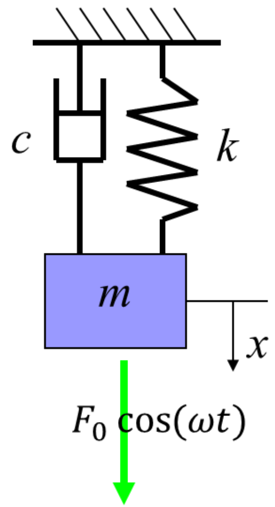
Measured response amplitude is defined as $\left(\frac{\delta_{st}}{1-(\omega^2/\omega_n^2)}\right) = 0.1$

Solve for natural frequency $\omega_n = 66.23\ \mathrm{rad/s}$

Using $\omega_n = \sqrt{\frac{k}{m}}$ the mass is found to be $m = 0.45$ kg

Harmonic excitation: damped

Consider the usual spring-mass-damper system with applied force $F(t) = F_0 \cos(\omega t)$:



Damped system under harmonic excitation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

The equation can also be expressed as:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)$$

- ❖ Natural frequency $\omega_n = \sqrt{k/m}$
- ❖ Normalized force amplitude $f_0 = \frac{F_0}{m}$
- ❖ Damping ratio $\zeta = \frac{c}{2\omega_n m}$

Harmonic excitation: damped

The damped system subject to harmonic excitation:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)$$

❖ The solution of the equation has two parts:

$$x(t) = (\text{complementary solution}) + \{\text{particular solution}\}$$

- A complementary solution for the homogenous equation for $F(t) = 0$
- A particular solution irrespective of the free damped-vibration for $F(t) = F_0 \cos(\omega t)$. This is also steady state response

❖ For $F(t) = 0 \Longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

- Complementary solution will depend on the damping. For the underdamped case, it has the form

$$x_C(t) = Ae^{-\zeta\omega_n t}\{\sin(\omega_d t + \phi)\}$$

- A and ϕ will be determined from the initial conditions x_0 and v_0 later

Harmonic excitation: damped

- For $F(t) = F_0 \cos(\omega t) \Longrightarrow \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)$
- The particular solution is assumed to be related to the forcing function:

$$x_P(t) = X\cos(\omega t - \theta)$$

  – Frequency will be the same with forcing function frequency except there could be a phase shift θ

  – Magnitude of response may be different from the forcing function and X is the amplitude of the steady state response

$$\dot{x}_P(t) = -\omega X \sin(\omega t - \theta) \quad \text{and} \quad \ddot{x}_P(t) = -\omega^2 X \cos(\omega t - \theta)$$

Substituting these into $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \cos(\omega t)$:

$$-\omega^2 X\cos(\omega t - \theta) - 2\zeta\omega_n\omega X\sin(\omega t - \theta) + \omega_n^2 X\cos(\omega t - \theta) = f_0\cos(\omega t)$$

$$(\omega_n^2 - \omega^2)X\cos(\omega t - \theta) - 2\zeta\omega_n\omega X\sin(\omega t - \theta) = f_0\cos(\omega t)$$

Harmonic excitation: damped

$$(\omega_n^2 - \omega^2) X \cos(\omega t - \theta) - 2\zeta\omega_n\omega X \sin(\omega t - \theta) = f_0 \cos(\omega t)$$

- Note:

$$\cos(\omega t - \theta) = \cos(\omega t)\cos(\theta) + \sin(\omega t)\sin(\theta)$$
$$\sin(\omega t - \theta) = \sin(\omega t)\cos(\theta) - \cos(\omega t)\sin(\theta)$$

- Substitute these into the above:

$$(\omega_n^2 - \omega^2) X \cos(\omega t)\cos(\theta) + (\omega_n^2 - \omega^2) X \sin(\omega t)\sin(\theta)$$
$$-2\zeta\omega_n\omega X \sin(\omega t)\cos(\theta) + 2\zeta\omega_n\omega X \cos(\omega t)\sin(\theta) = f_0 \cos(\omega t)$$

$$\{(\omega_n^2 - \omega^2) X \cos(\theta) + 2\zeta\omega_n\omega X \sin(\theta)\}\cos(\omega t)$$
$$+ \{(\omega_n^2 - \omega^2) X \sin(\theta) - 2\zeta\omega_n\omega X \cos(\theta)\}\sin(\omega t) = f_0 \cos(\omega t)$$

- This can be separated into 2 equations:

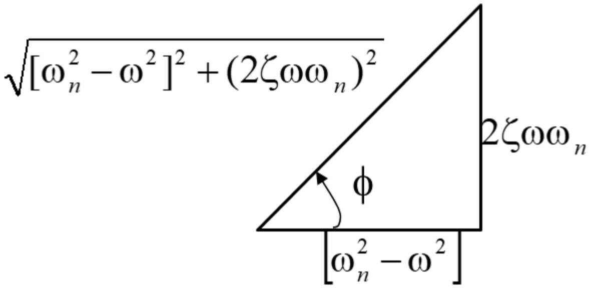
$$\{(\omega_n^2 - \omega^2) X \cos(\theta) + 2\zeta\omega_n\omega X \sin(\theta)\}\cos(\omega t) = f_0 \cos(\omega t)$$
$$\{(\omega_n^2 - \omega^2) X \sin(\theta) - 2\zeta\omega_n\omega X \cos(\theta)\}\sin(\omega t) = 0$$

Harmonic excitation: damped

$$\{(\omega_n^2 - \omega^2)X\cos(\theta) + 2\zeta\omega_n\omega X\sin(\theta)\} = f_0$$
$$\{(\omega_n^2 - \omega^2)X\sin(\theta) - 2\zeta\omega_n\omega X\cos(\theta)\}\sin(\omega t) = 0$$

- Using the second equation:

$$\{(\omega_n^2 - \omega^2)X\sin(\theta) - 2\zeta\omega_n\omega X\cos(\theta)\} = 0$$
$$(\omega_n^2 - \omega^2)\ \sin(\theta) = 2\zeta\omega_n\omega\cos(\theta)$$
$$\tan\theta = \frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)} \quad \text{or} \quad \theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)}$$



- Substitute cosine and sine from the triangle into the first equation:

$$\left\{\frac{(\omega_n^2 - \omega^2)X(\omega_n^2 - \omega^2)}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} + \frac{2\zeta\omega_n\omega X(2\zeta\omega_n\omega)}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\right\} = f_0$$

$$\left\{\frac{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\right\}X = f_0 \quad \text{or} \quad X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

❖ The particular(or steady state) solution is $x_P(t) = X\cos(\omega t - \theta)$

Harmonic excitation: damped

- The total solution for the harmonically excited underdamped system is

$$x(t) = x_C(t) + x_p(t) = Ae^{-\zeta\omega_n t}\{\sin(\omega_d t + \phi)\} + X\cos(\omega t - \theta)$$

Note that A and ϕ need to be determined from the initial conditions x_0 and v_0

- The total solution for the harmonically excited critically damped system is

$$x(t) = x_C(t) + x_p(t) = e^{-\omega_n t}(a_1 + a_2 t) + X\cos(\omega t - \theta)$$

Note that a_1 and a_2 need to be determined from the initial conditions x_0 and v_0

- The total solution for the harmonically excited over damped system is

$$x(t) = a_1 e^{\omega_n\left(-\zeta - \sqrt{\zeta^2 - 1}\right)t} + a_2 e^{\omega_n\left(-\zeta + \sqrt{\zeta^2 - 1}\right)t} + X\cos(\omega t - \theta)$$

Note that a_1 and a_2 need to be determined from the initial conditions x_0 and v_0

Example 2

Given a spring-mass-damper system has spring constant k = 4000 N/m, mass m = 10 kg, and damping coefficient c = 40 Ns/m. Find the steady state and total responses of the system under the harmonic force $F(t) = 200\sin(10t)$ given the initial conditions x_0 = 0 and v_0 = 10 m/s.

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20$ rad/s;
- Damping ratio is $\zeta = \frac{c}{2\omega_n m} = \frac{40}{2(20)(10)} = 0.1$ (system is underdamped)
- Force is $F(t) = 200\sin(10t)$, i.e. $F_0 = 200$, and $f_0 = \frac{F_0}{m} = \frac{200}{10} = 20$, with excitation frequency $\omega = 10$ rad/s;
- $X = \frac{f_0}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} = 0.066$ m; and $\theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)} = 0.1326$ rad.;
- Hence steady state $x_P(t) = X\sin(\omega t - \theta) = 0.066\sin(10t - 0.1326)$

Example 2

- ❖ The total solution for the harmonically excited underdamped system is

$$x(t) = x_C(t) + x_p(t) = Ae^{-\zeta\omega_n t}\{\sin(\omega_d t + \phi)\} + X\sin(\omega t - \theta)$$

- Damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2} = 19.9$ rad/s

$$x(t) = Ae^{-2t}\{\sin(19.9t + \phi)\} + 0.066\sin(10t - 0.1326)$$

$$\dot{x}(t) = -2Ae^{-2t}\{\sin(19.9t + \phi)\} + 19.9Ae^{-2t}\{\cos(19.9t + \phi)\} + 0.66\cos(10t - 0.1326)$$

- ▪ For $t = 0$, $x = x_0 = 0$ and $\dot{x} = v_0 = 10$ m/s:

$$0 = A\{\sin(\phi)\} + 0.066\sin(-0.1326)$$

$$10 = -2A\{\sin(\phi)\} + 19.9A\{\cos(\phi)\} + 0.66\cos(-0.1326)$$

Simplify:

$$0 = A\{\sin(\phi)\} - 0.008726 \quad \text{or} \quad A\{\sin(\phi)\} = 0.008726$$

$$10 = -2A\{\sin(\phi)\} + 19.9A\{\cos(\phi)\} + 0.654$$

Substitute $A\{\sin(\phi)\} = 0.008726$ into second equation:

$$10 = -2 \times 0.008726 + 19.9A\{\cos(\phi)\} + 0.654$$

$$A\{\cos(\phi)\} = 0.47$$

Therefore $\tan\phi = \frac{0.008726}{0.47}$ or $\phi = 0.0185$ rad

Example 2

- $A\{\sin(\phi)\} = 0.008726$ and $A\{\cos(\phi)\} = 0.47$

$$A = \sqrt{(A\sin(\phi))^2 + (A\cos(\phi))^2} = 0.47$$

The total solution is:

$$x(t) = Ae^{-2t}\{\sin(19.9t + \phi)\} + 0.066\sin(10t - 0.1326)$$

$$x(t) = 0.47e^{-2t}\{\sin(19.9t + 0.0185)\} + 0.066\sin(10t - 0.1326)$$

Steady state response

- For the underdamped, critically damped and over damped cases, the transient response will diminish to zero. Hence, focus is on the steady state response $X\cos(\omega t - \theta)$
- The damped system subject to harmonic excitation:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0\cos(\omega t)$$

In the steady state:

$$x(t) = X\cos(\omega t - \theta)$$

where

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{F_0}{\sqrt{[k - m\omega^2]^2 + (c\omega)^2}}$$

$$\theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2)} = \tan^{-1}\frac{c\omega}{[k - m\omega^2]}$$

Steady state response

The amplitude and phase angle is also often written in non-dimensional form to show that they are function of frequency ratio r = (ω/ω_n):

$$X = \frac{\frac{F_0}{k}}{\sqrt{\left[1-\frac{m\omega^2}{k}\right]^2+\left(\frac{c\omega}{k}\right)^2}} \qquad \text{and} \qquad \theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)}$$

Define amplitude ratio as

$$M = \frac{X}{\delta_{st}} = \frac{kX}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} = \frac{1}{\sqrt{[1-r^2]^2+[2\zeta r]^2}}$$

Phase shift: $\theta = \tan^{-1}\frac{2\zeta\omega_n\omega}{(\omega_n^2-\omega^2)} = \tan^{-1}\frac{2\zeta r}{[1-r^2]}$

Example 3

A simple spring–mass–damper system with mass $m = 49.2 \times 10^{-3}$ kg, damping coefficient $c = 0.11$ kg/s, and spring constant $k = 857.8$ N/m is subjected to a harmonic force of magnitude $F_0 = 0.492$ N at a forcing frequency of $\omega = 132$ rad/s. Determine the amplitude ratio, phase shift, the steady state response amplitude, and the static deflection

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{857.8}{49.2 \times 10^{-3}}} = 132$ rad/s;
- Damping ratio is $\zeta = \frac{c}{2\omega_n m} = \frac{0.11}{2(132)(49.2 \times 10^{-3})} = 0.0085$
- $f_0 = \frac{F_0}{m} = \frac{0.492}{49.2 \times 10^{-3}} = 10$ N/m

Example 3

- At excitation frequency $\omega = 132$ rad/s; frequency ratio $= \frac{\omega}{\omega_n} = 1$ rad/s;
- Steady state response amplitude $X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = 0.034$ m;
- Phase shift $\theta = \tan^{-1}\frac{2\zeta r}{[1-r^2]} = 1.571$ rad.;
- Amplitude ratio $M = \frac{1}{\sqrt{[1-r^2]^2 + [2\zeta r]^2}} = 59$
- Static deflection $\delta_{st} = (F_0/k) = 0.000574$

Note resonance $\omega = \omega_n$ and the steady state amplitude is 59 times the static deflection