ME1020
Mechanical vibrations

Lecture 3

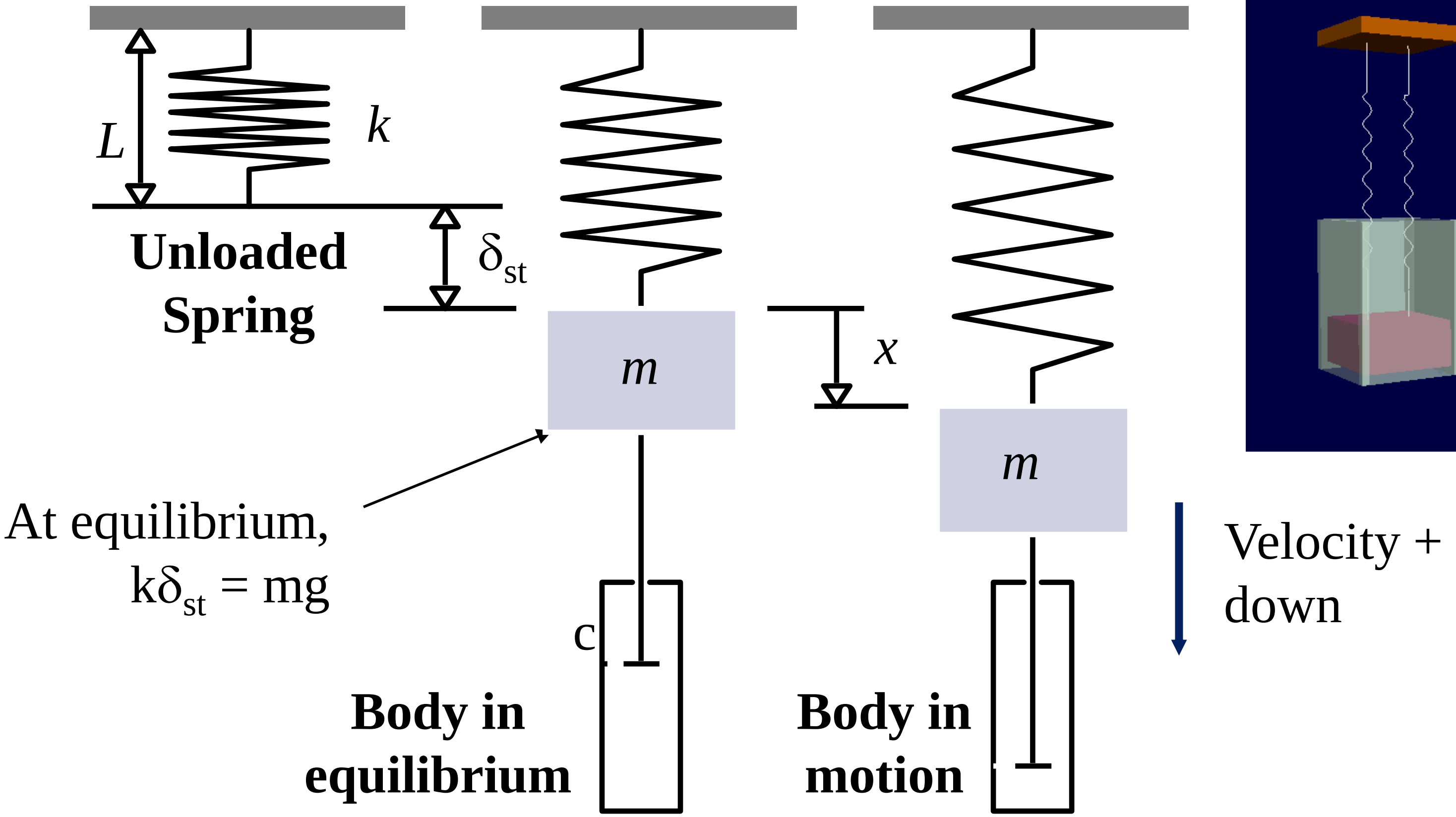
Free vibration (damped 1DOF system)

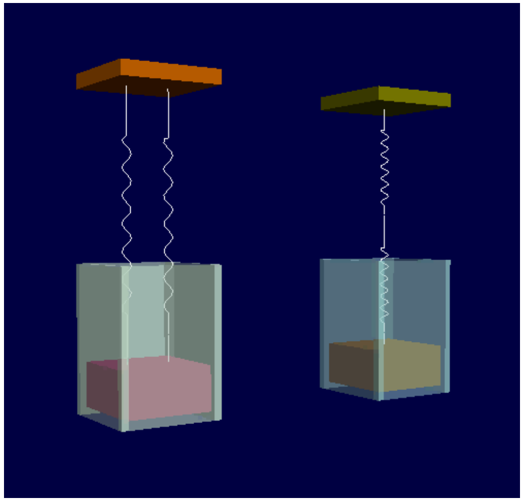
Objectives

- Describe the characteristics of viscous dampers
- Derive the 1DOF free damped vibration system model based on Newton's laws and Lagrange's equation
- Determine the damping ratio, and response of 1DOF free damped vibration system responses

Friction and damping

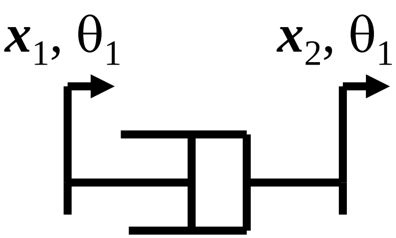
All vibrations have damping to some degree due to dry friction





Friction and damping

Viscous damping



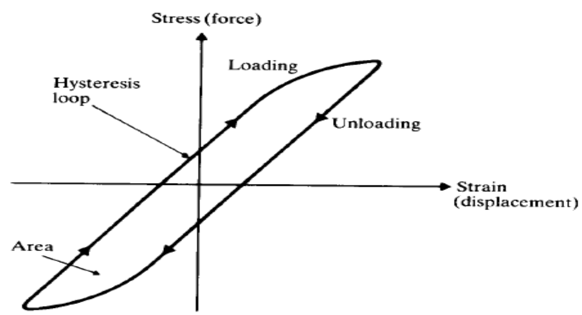
Translational

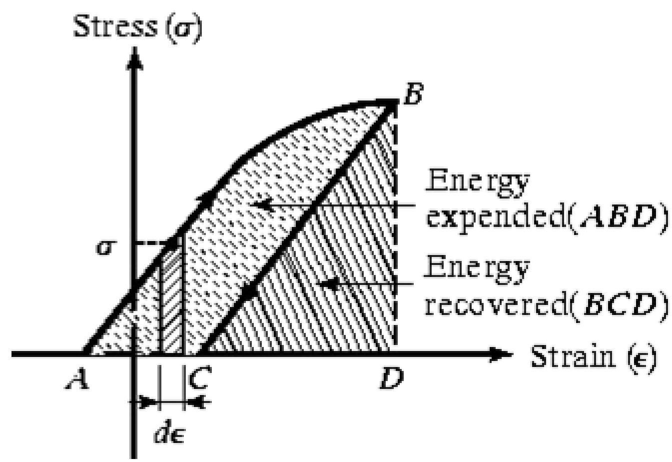
$$F = c\dot{x}$$

Torsional

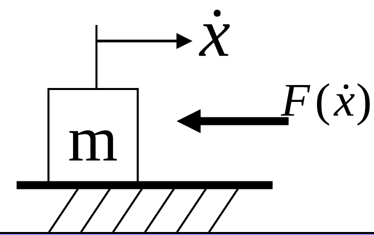
$$T = c\dot{\theta}$$

Hysteretic damping





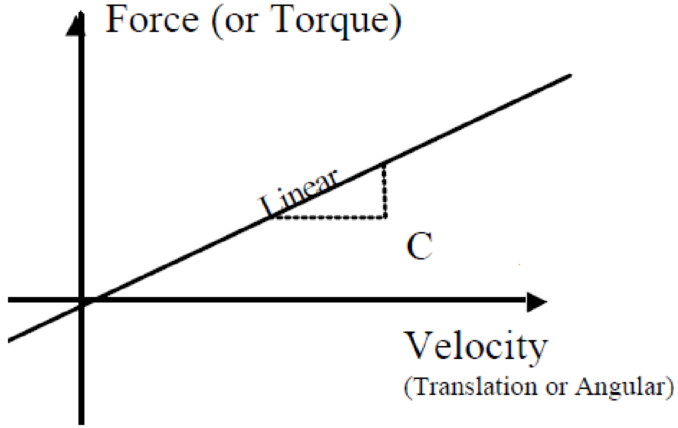
Coulomb damping



$$F(\dot{x}) = \mu mg\,\mathrm{sgn}(\dot{x})$$

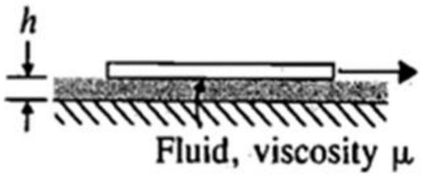
Viscous damper

TYPE	LOAD
Translational damper	$F = c\dot{x}$
Rotational spring	$T = c\dot{\theta}$



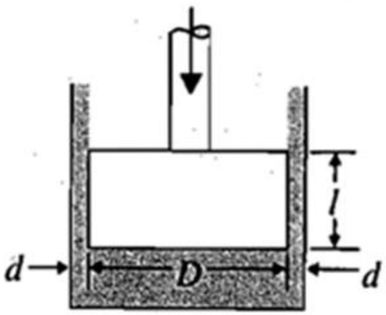
- Force F
- Torque T
- Damping coefficient c
- Rate of linear deformation $\dot{x}$
- Rate of angular deformation $\dot{\theta}$

Viscous damper



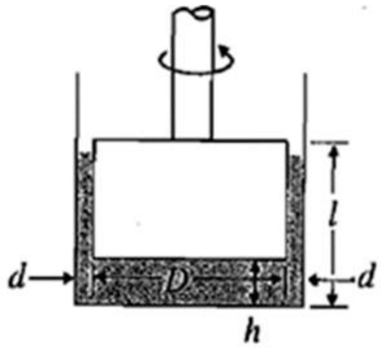
Relative motion between parallel surfaces
(A = area of smaller plate)

$$c_{eq} = \frac{\mu A}{h}$$



Dashpot (axial motion of a piston in a cylinder)

$$c_{eq} = \mu \frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)$$



Torsional damper

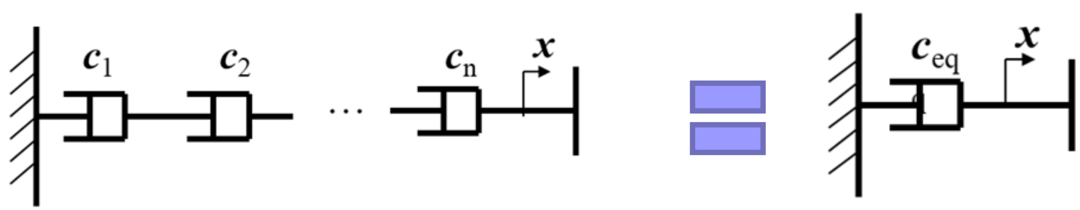
$$c_{eq} = \frac{\pi\mu D^2(l-h)}{2d} + \frac{\pi\mu D^3}{32h}$$

μ = fluid viscosity

Viscous damper

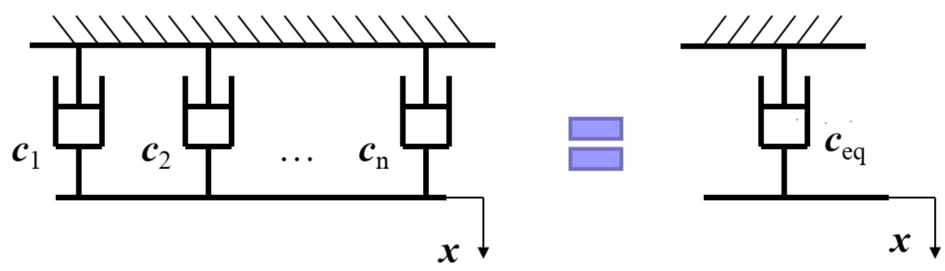
For n dampers (with coefficients $c_1, c_2, \cdots, c_n$) connected in series, the equivalent damper coefficient is:

$$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \ldots + \frac{1}{c_n}$$



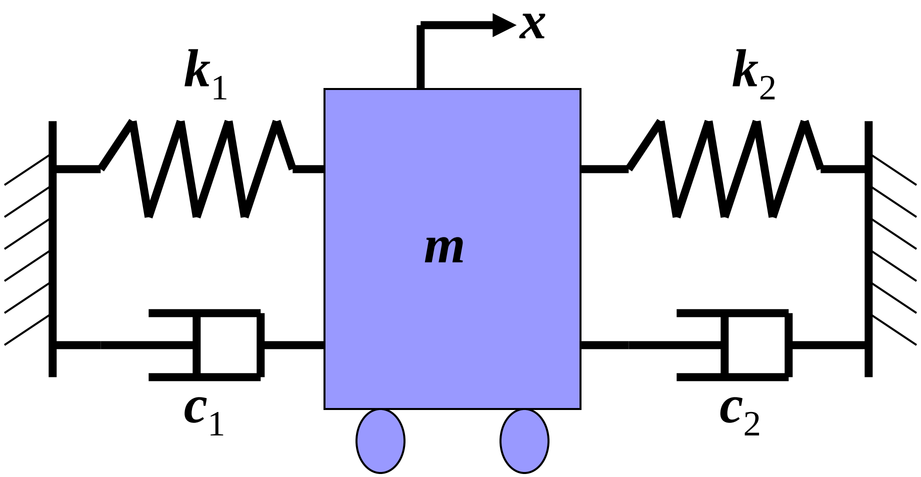
For n dampers (with coefficients $c_1, c_2, \cdots, c_n$) connected in parallel, the equivalent damper coefficient is:

$$c_{eq} = c_1 + c_2 + \cdots + c_n$$

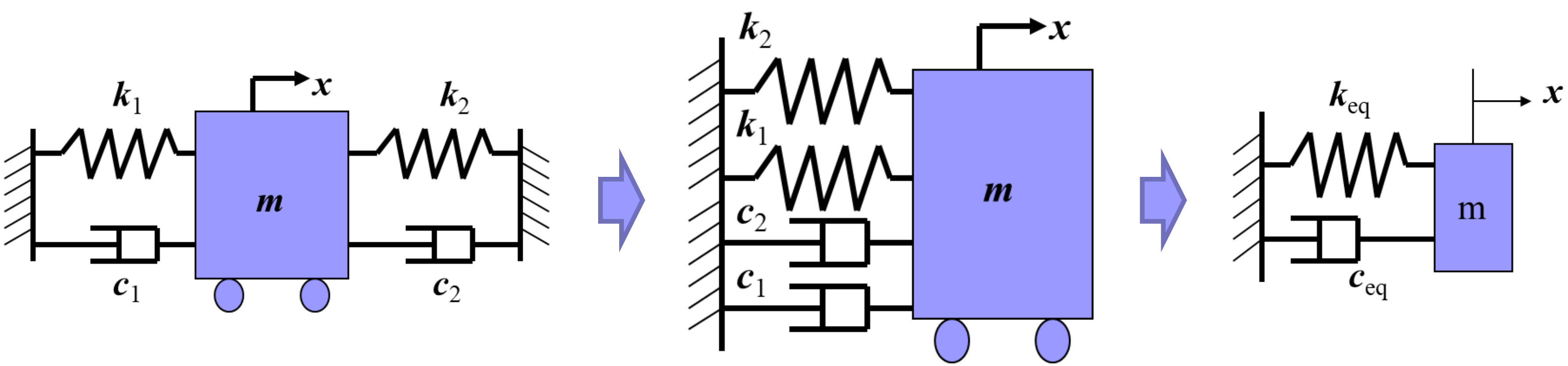


Example 1

Represent the given system as an equivalent vibratory system with mass m, equivalent stiffness k_{eq}, and equivalent damping c_{eq}.



Example 1



Equivalent spring constant: $k_{eq} = k_1 + k_2$

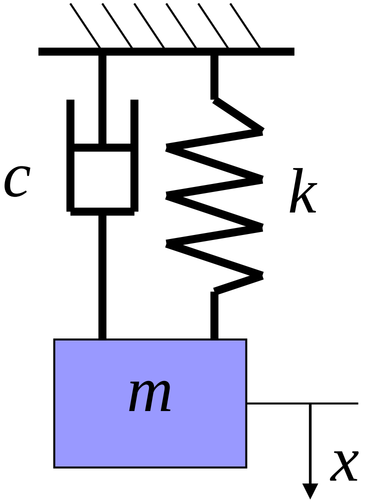
Equivalent damper coefficient: $c_{eq} = c_1 + c_2$

The system equation is

$$m\ddot{x} + c_{eq}\dot{x} + k_{eq}x = 0$$

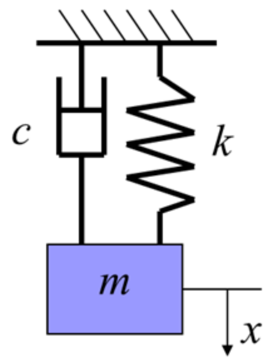
Spring-mass-damper system

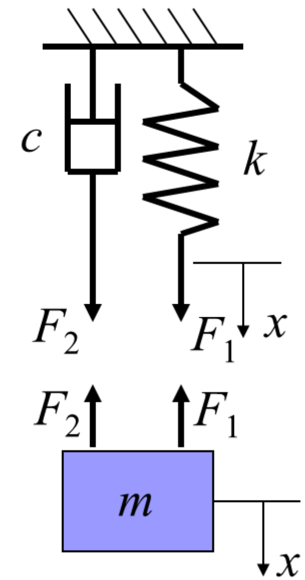
An example of a structure that can be idealized as simple spring-mass-damper system (if friction is not negligible):



Equivalent spring-mass-damper system

Spring-mass-damper system





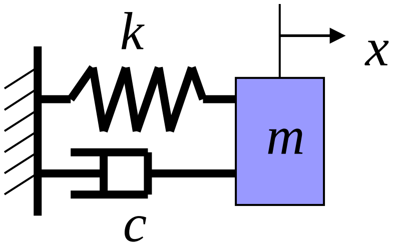
- At the equilibrium position, all the forces are balanced and the system is stationary
- Draw the free-body diagram about the equilibrium position
- The spring force is $F_1 = kx$
- The damper force is $F_2 = c\dot{x}$
- Applying Newton's law on the mass

$$m\ddot{x} = -F_1 - F_2 = -kx - c\dot{x}$$

- The spring-mass equation is

$$m\ddot{x} + c\dot{x} = kx = 0$$

Spring-mass-damper system



- A mass-spring-damper system subjected to initial conditions x_0 and v_0 is an example of a single-degree of freedom "free vibration" damped system
- The system equation has the form:

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad \text{or} \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}}$
- Damping ratio $\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} = \frac{c\omega_n}{2k}$

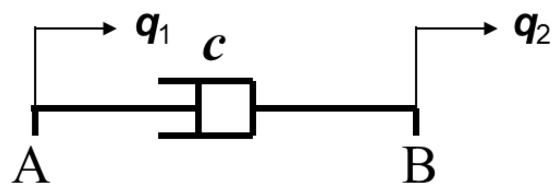
Lagrange's equation

In terms of generalized coordinate q, the Lagrange's equation for a single DOF free damped system has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

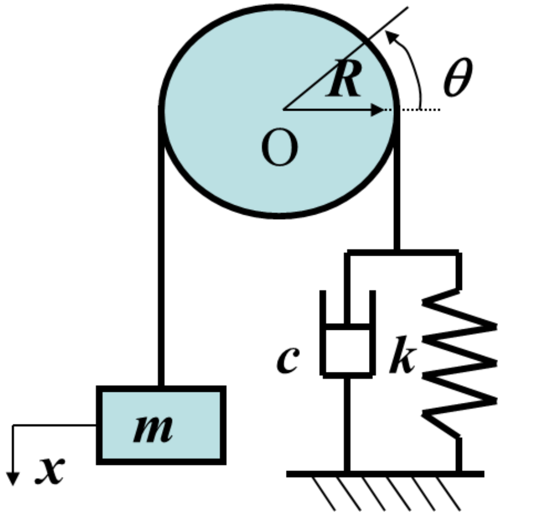
- T = Kinetic energy
- U = Potential energy
- D = Rayleigh's damping (or dissipation) function
- q = generalized coordinate that completely describe the dynamical system

$$D = \frac{1}{2}c(\dot{q}_2 - \dot{q}_1)^2$$



Example 2

Use Lagrange's equation to derive the equation of motion for the system using generalized coordinate θ. The mass moment of inertia of the disk about "O" is I.



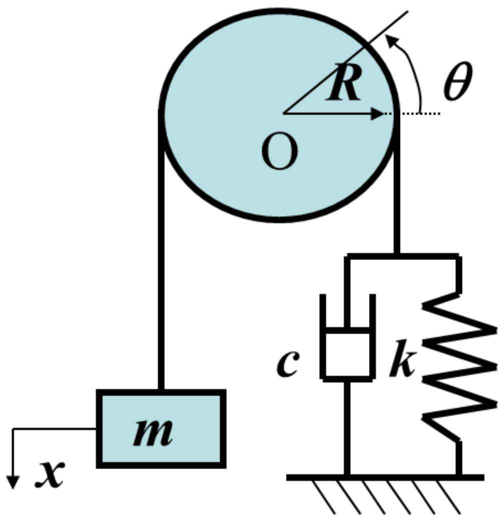
- Note that $x = R\theta$
- Generalized coordinate $q = \theta$
- Kinetic energy $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} + mR^2\dot{\theta}$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial \theta} = 0$$

Example 2



- Dissipation function $D = \frac{1}{2}c\dot{x}^2 = \frac{1}{2}cR^2\dot{\theta}^2$

$$\frac{\partial D}{\partial \dot{q}} = \frac{\partial D}{\partial \dot{\theta}} = cR^2\dot{\theta}$$

- Potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}kR^2\theta^2$

$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial \theta} = kR^2\theta$$

- Apply Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = 0$$

$$I\ddot{\theta} + mR^2\ddot{\theta} + cR^2\dot{\theta} + kR^2\theta = 0$$

$$(I + mR^2)\ddot{\theta} + cR^2\dot{\theta} + kR^2\theta = 0$$

1DOF free damped

A 1DOF free damped system equation has the form:

$$m\ddot{x} + c\dot{x} + kx = 0$$

- ❖ System equation can be written in the form:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\, x = 0$$

- Natural frequency $\omega_n = \sqrt{(k/m)}$
- Damping ratio $\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} = \frac{c\omega_n}{2k}$

- ❖ When $\zeta = 0$, system is undamped and the response is harmonic motion
- ❖ When $\zeta = 1$, system is critically damped with $c = c_{cr} = 2\sqrt{km} = 2m\omega_n$ and c_{cr} is called the critical damping coefficient

Note: the damping ration can be defined as $\zeta = c/c_{cr}$

- ❖ When $0 < \zeta < 1$, the system is underdamped
- ❖ When $\zeta > 1$, the system is over-damped

The responses for under, over and critically damped cases are different

1DOF free damped

- If the DOF free damped system is vibrating then something must have (in the past) transferred energy into to the system and caused it to move
- For example the mass could have been moved a distance x_0 and then released at $t = 0$ (i.e. given Potential energy) or given an initial velocity v_0 (i.e. given Kinetic energy) or some combination of the two cases. These are called initial conditions
- The solution to $m\ddot{x} + c\dot{x} + kx = 0$ is assumed to have the form $x = Ae^{\lambda t}$
- Substitute this back into the governing equation:

$$(m\lambda^2 + c\lambda + k)Ae^{\lambda t} = 0$$

- This is only satisfied for: $m\lambda^2 + c\lambda + k = 0$
- The solution for λ (or the root of the characteristics equation) is:

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Underdamped case

System equation can be written in the form: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\, x = 0$

- Underdamped case $0 < \zeta < 1$,

$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d$ will result in 2 complex roots

- $\omega_d = \omega_n\sqrt{1-\zeta^2}$ = damped natural frequency
- Solution is of the form:

$$x(t) = Ae^{\lambda t} = e^{-\zeta\omega_n t}\left(a_1 e^{j(\omega_d t)} + a_2 e^{-j(\omega_d t)}\right)$$

$$x(t) = e^{-\zeta\omega_n t}\{B\sin(\omega_d t + \phi)\}$$

$$\dot{x}(t) = -(\zeta\omega_n)e^{-\zeta\omega_n t}\{B\sin(\omega_d t + \phi)\} + e^{-\zeta\omega_n t}\{B\omega_d\cos(\omega_d t + \phi)\}$$

- At time $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$:

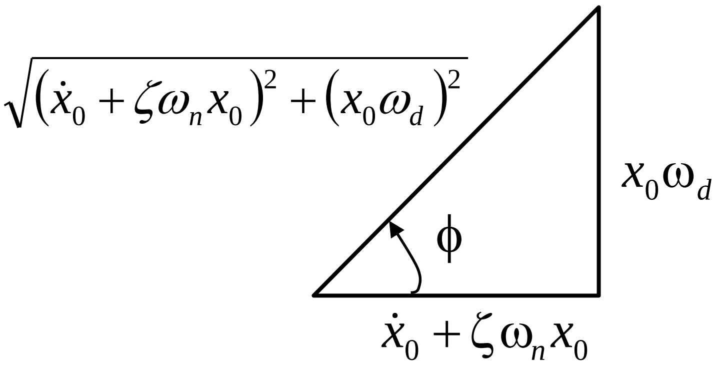
$$x(0) = x_0 = B\sin(\phi)$$

$$\dot{x}(0) = v_0 = -(\zeta\omega_n)\{B\sin(\phi)\} + \{B\omega_d\cos(\phi)\}$$

$$\frac{v_0}{x_0} = -\zeta\omega_n + \frac{(\omega_d)}{\tan(\phi)} \quad \text{or } \tan(\phi) = \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}$$

Underdamped case

Triangle for $\tan(\phi) = \frac{x_0\omega_d}{\dot{x}_0+\zeta\omega_n x_0} \Longrightarrow$



- ❖ From the right-angle triangle:

$$\sin\phi = \frac{x_0\omega_d}{\sqrt{(\dot{x}_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}$$

- ❖ Since $x_0 = B\sin(\phi)$

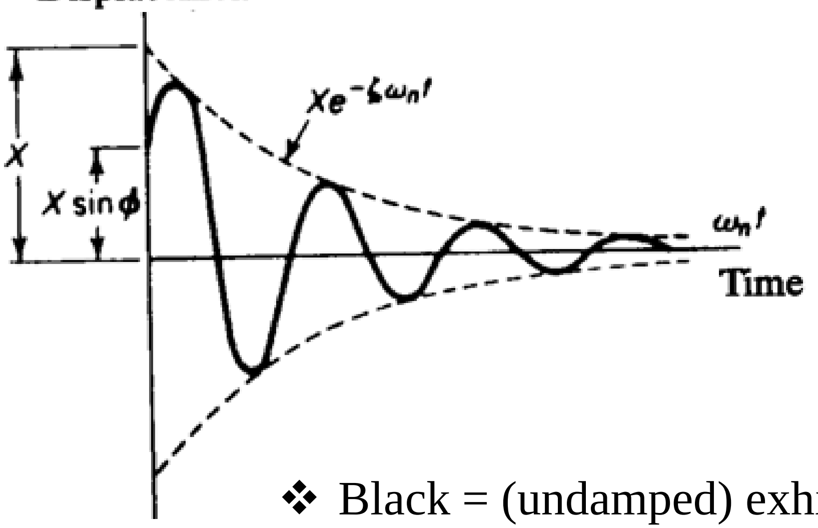
$$B = \frac{\sqrt{(\dot{x}_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d}$$

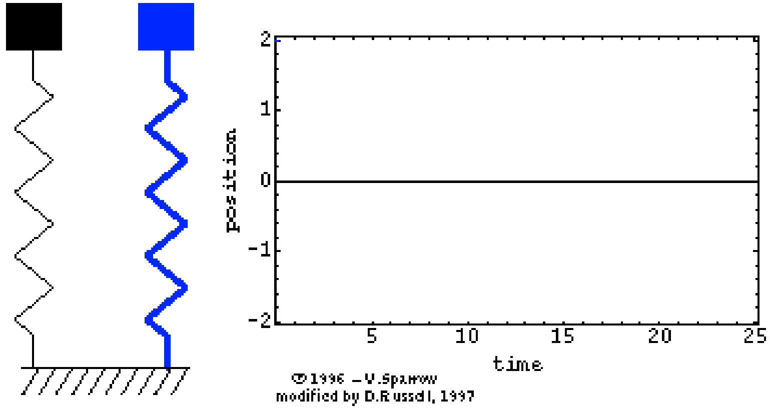
- ❖ With B and ϕ the complete equation is obtained:

$$x(t) = e^{-\zeta\omega_n t}\{B\sin(\omega_d t + \phi)\}$$

Underdamped case

$$x(t) = e^{-\zeta\omega_n t}\{B\sin(\omega_d t + \phi)\} \begin{cases} \phi = \tan^{-1}\left(\dfrac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right) \\ B = \dfrac{\sqrt{(\dot{x}_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d} \end{cases}$$





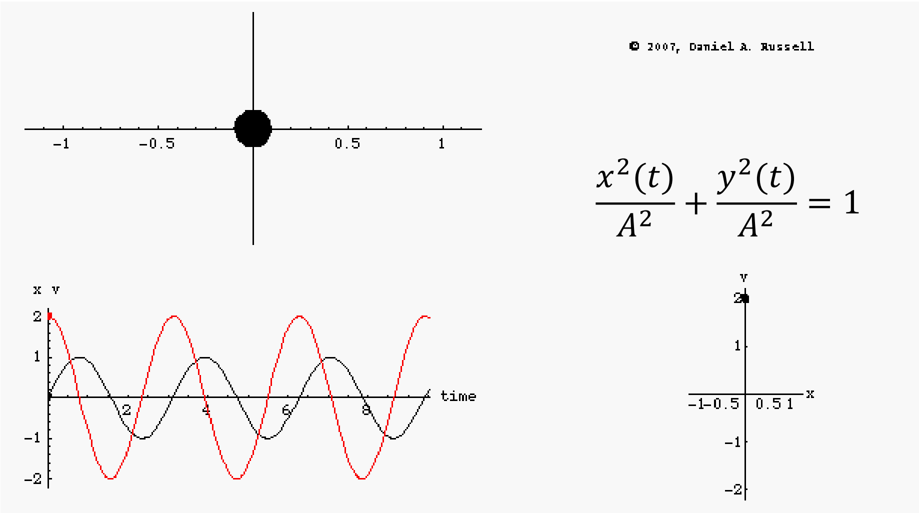
❖ Black = (undamped) exhibits SHM

❖ Blue = (underdamped) exhibits decaying oscillatory motion

Phase plane comparison:

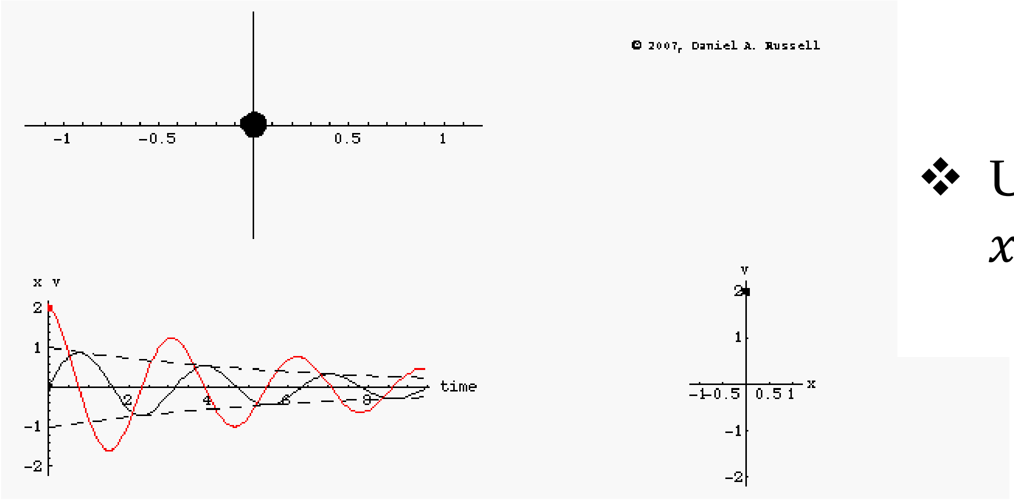
- Plot of x with y for same system with same initial conditions x_0 and v_0 where

$$y = \frac{\dot{x}(t)}{\omega_n}$$



❖ Undamped case

$$x(t) = A\sin(\omega_n t + \phi)$$



❖ Underdamped case

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

Underdamped case

- Rate of natural logarithm decay depends on damping
- Damped natural period τ_d is time between successive peaks and is related to damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \text{ by}$$

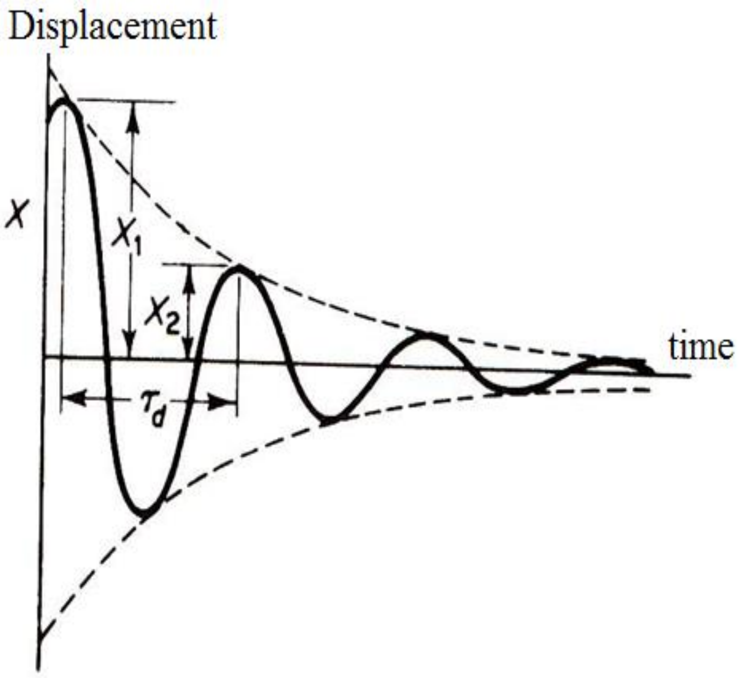
$$\tau_d = \frac{2\pi}{\omega_d}$$

- For peaks n cycles apart, logarithmic decrement δ is

$$\delta = \frac{1}{n} \ln \frac{X_1}{X_{n+1}} = \frac{2\pi\zeta}{\sqrt{1 - \zeta^2}}$$

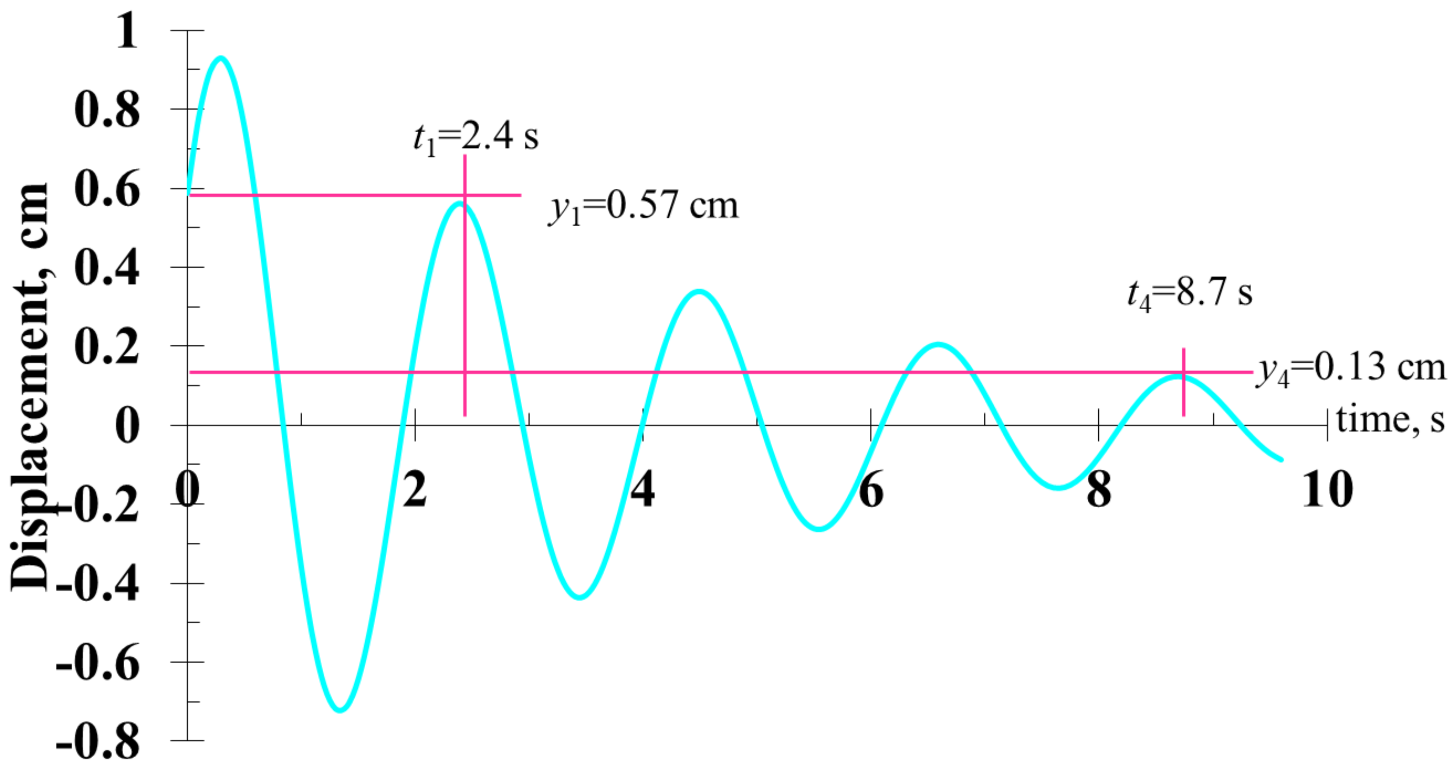
- Damping ratio can be found from δ by

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$



Example 3

The following free response data were obtained from a vibrating system Find the natural frequency, damped natural frequency and the damping ratio.



Critically damped case

System equation can be written in the form: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\, x = 0$

- Critically damped case $\zeta = 1$,

  $\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\omega_n$ will result in 2 repeated real roots

- Solution is of the form (with 2 independent solutions):

$$x(t) = Ae^{\lambda t} = e^{-\omega_n t}(a_1 + a_2 t)$$

$$\dot{x}(t) = -a_1\omega_n e^{-\omega_n t} - a_2\omega_n t e^{-\omega_n t} + a_2 e^{-\omega_n t}$$

- At time $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$x(0) = x_0 = a_1$$

$$\dot{x}(0) = v_0 = -a_1\omega_n + a_2 = -x_0\omega_n + a_2$$

$$a_2 = v_0 + x_0\omega_n$$

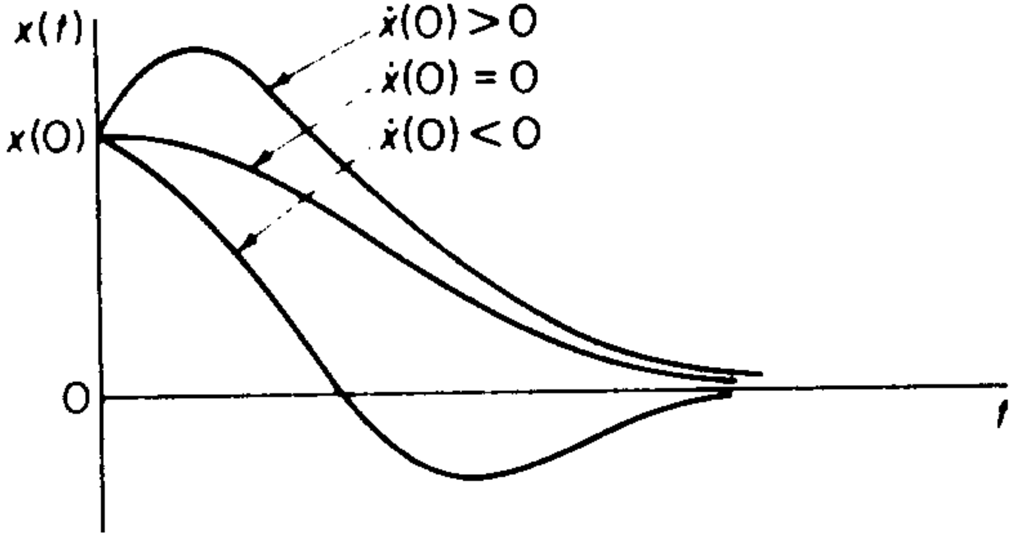
- The response is

$$x(t) = (x_0 + x_0\omega_n t + v_0 t)e^{-\omega_n t}$$

Critically damped case

$$x(t) = (x_0 + x_0\omega_n t + v_0 t)e^{-\omega_n t}$$

- Exponential decay: fastest approach to steady state without oscillation



Overdamped case

System equation can be written in the form: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2\, x = 0$

- Over damped case $\zeta > 1$, there are 2 distinct real roots

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$$

Solution is of the form:

$$x(t) = Ae^{\lambda t} = a_1 e^{\omega_n\left(-\zeta-\sqrt{\zeta^2-1}\right)t} + a_2 e^{\omega_n\left(-\zeta+\sqrt{\zeta^2-1}\right)t}$$

$$\dot{x}(t) = a_1\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} + a_2\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$$

- At time $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$: $\quad x(0) = x_0 = a_1 + a_2$

$$\dot{x}(0) = v_0 = a_1\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n + a_2\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$v_0 = -\zeta\omega_n(a_1 + a_2) + \left(\omega_n\sqrt{\zeta^2 - 1}\right)(-a_1 + a_2)$$

$$v_0 + \zeta\omega_n x_0 = \left(\omega_n\sqrt{\zeta^2 - 1}\right)(-a_1 + a_2)$$

Overdamped case

$$(-a_1 + a_2) = \frac{v_0 + \zeta\omega_n x_0}{\omega_n\sqrt{\zeta^2 - 1}}$$

$$a_1 + a_2 = x_0$$

- Solve for a_1 and a_2:

$$a_1 = \frac{-v_0 - \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

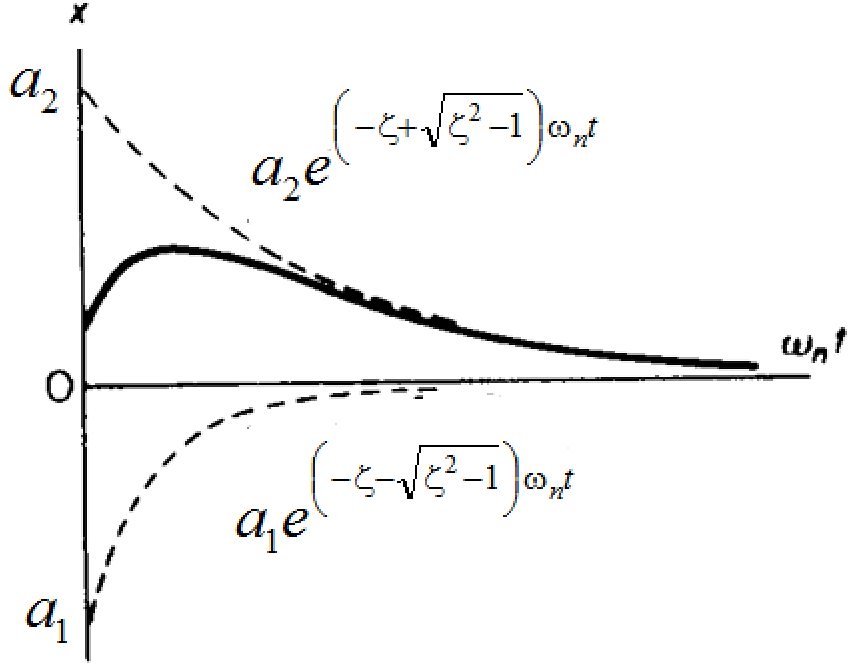
$$a_2 = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

- The response is:

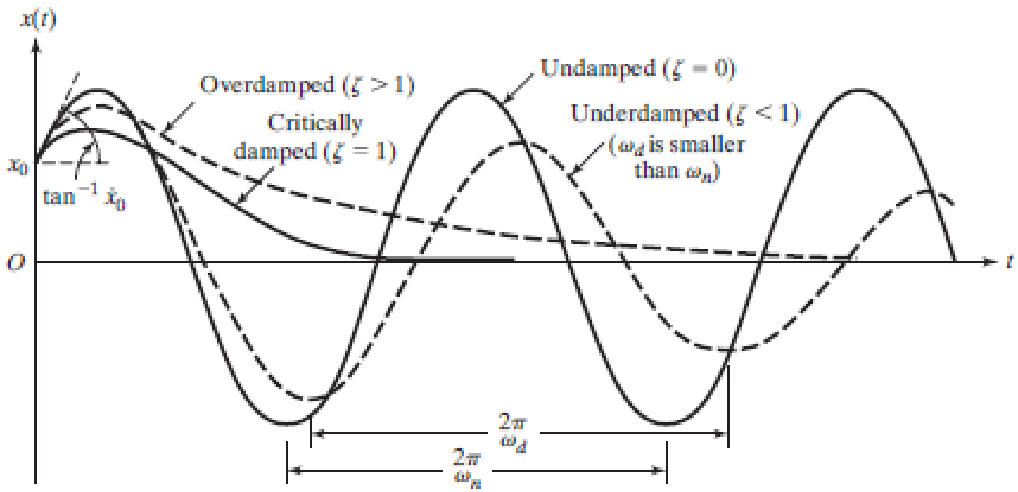
$$x(t) = a_1 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} + a_2 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

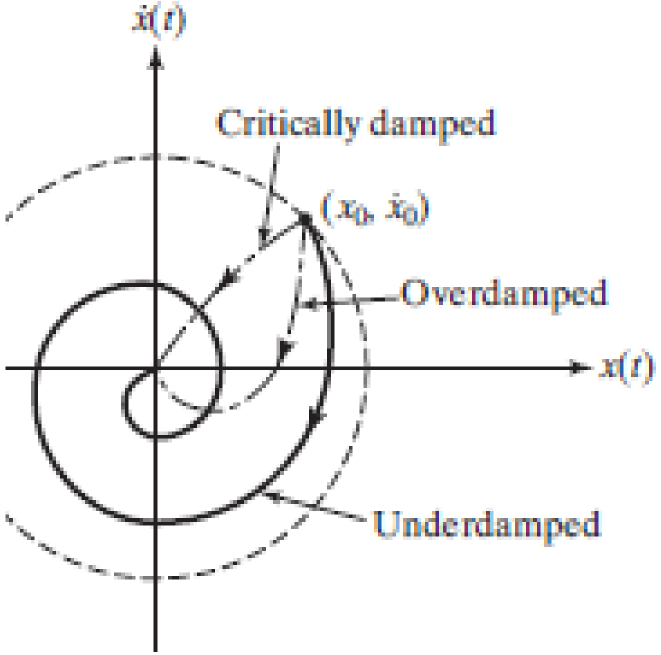
Overdamped case

$$x(t) = a_1 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} + a_2 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} \quad \begin{cases} a_1 = \dfrac{-v_0 - \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \\ a_2 = \dfrac{v_0 + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \end{cases}$$



Comparison





Time response characteristics

Phase plane representations