ME1020
Mechanical vibrations

Lecture 2

Free vibration (undamped 1DOF system)

Objectives

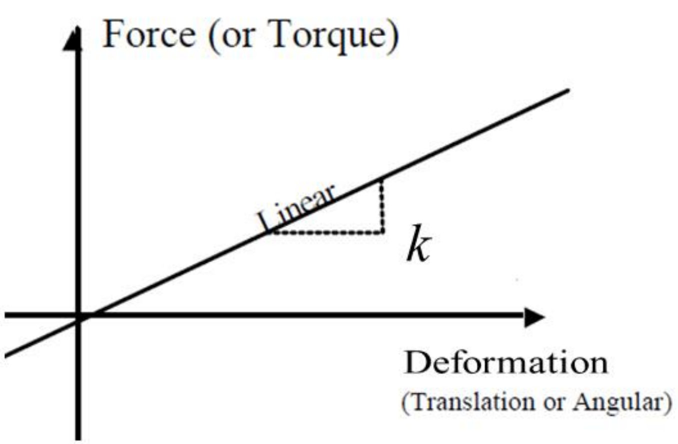
- Review of spring elements
- Derive the 1DOF free undamped vibration system model based on the energy approach and Lagrange's equation
- Determine the natural frequency, period, and response of 1DOF free undamped vibration system response

Review – spring elements

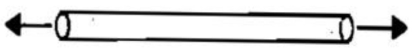
TYPE	LOAD	ENERGY
Translational spring	$F = kx$	$U = \frac{1}{2}kx^2$
Rotational spring	$T = k\theta$	$U = \frac{1}{2}k\theta^2$

Spring is an element associated with storage of potential energy

- Force F
- Torque T
- Spring constant k
- Linear deformation x
- Angular deformation θ
- Energy U

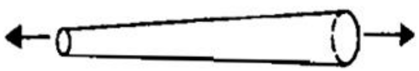


Review – spring elements



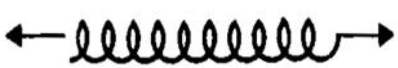
Rod under axial load
(l = length, A = cross sectional area)

$$k_{eq} = \frac{EA}{l}$$



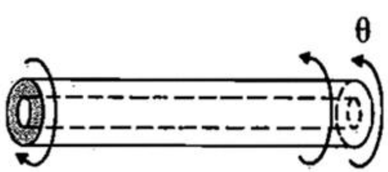
Tapered rod under axial load
(D, d = end diameters)

$$k_{eq} = \frac{\pi EDd}{4l}$$



Helical spring under axial load
(d = wire diameter, D = mean coil diameter, n = number of active turns)

$$k_{eq} = \frac{Gd^4}{8nD^3}$$

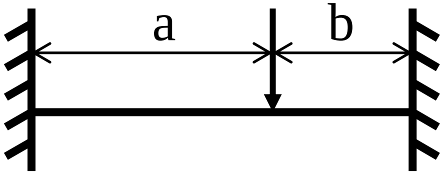


Hollow shaft under torsion
(l = length, D = outer diameter, d = inner diameter)

$$k_{eq} = \frac{\pi G}{32l}(D^4 - d^4)$$

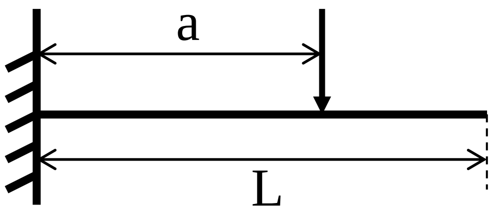
- E = Young's modulus
- G = shear modulus;

Review – spring elements



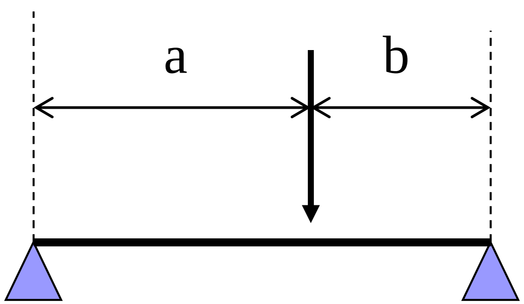
Fixed-fixed beam

$$k_{eq} = \frac{3EI(a+b)^3}{a^3 b^3}$$



Cantilever beam

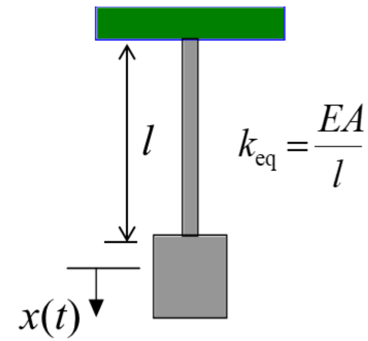
$$k_{eq} = \frac{3EI}{a^3}$$

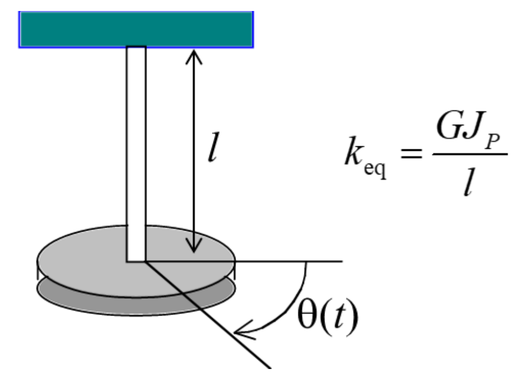


Simply supported beam

$$k_{eq} = \frac{3EI(a+b)}{a^2 b^2}$$

- ❖ E = Young's modulus
- ❖ I = Moment of inertia;



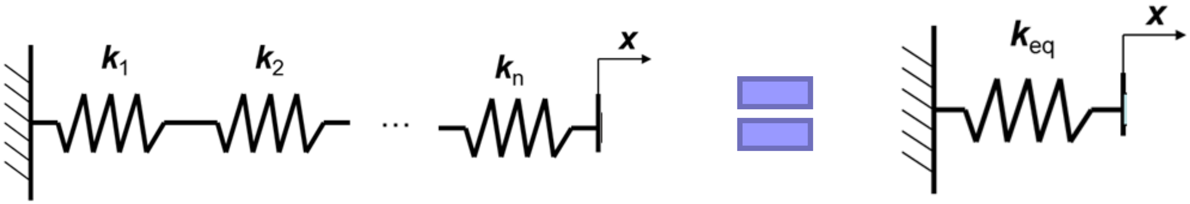


- ❖ J_p = polar area moment of inertia

Review – spring elements

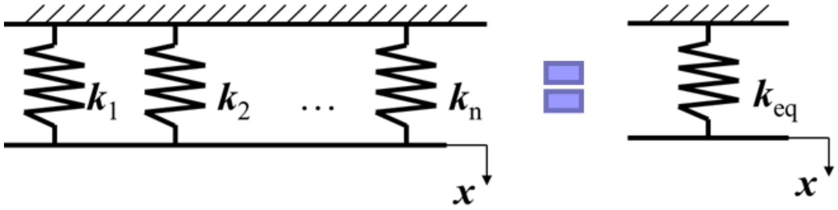
For n springs (with spring constants $k_1, k_2, \cdots, k_n$) connected in series, the equivalent spring constant is:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}$$



For n springs (with spring constants $k_1, k_2, \cdots, k_n$) connected in parallel, the equivalent spring constant is:

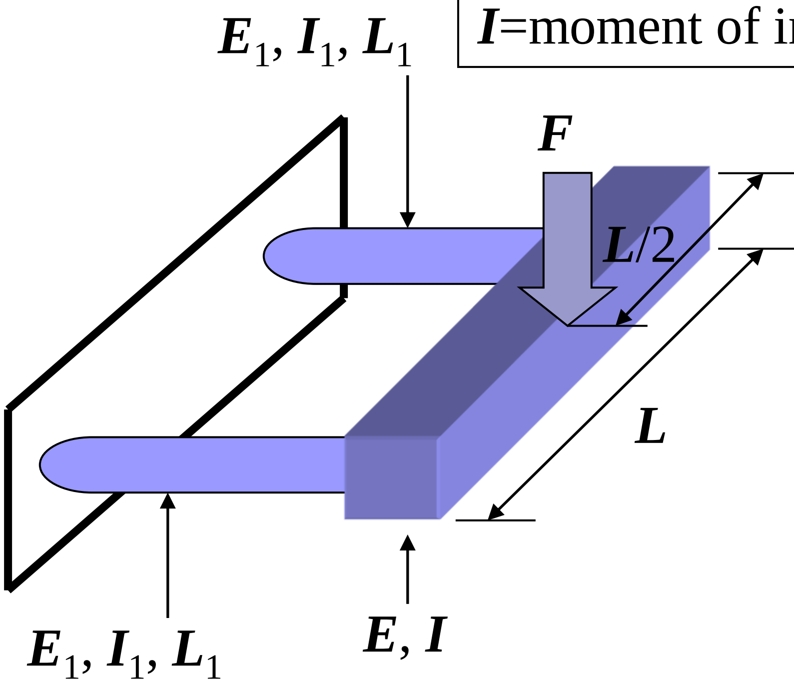
$$k_{eq} = k_1 + k_2 + \cdots + k_n$$

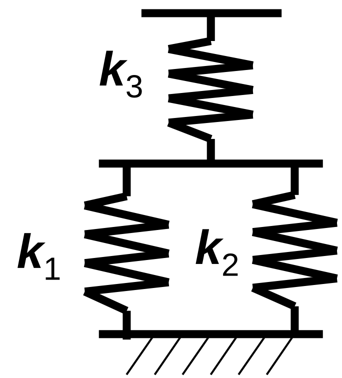


Example 1

The 2 circular beams are pinned to the ends of the rectangular beam. Find the equivalent spring constant for the transverse loading

E=Young's modulus;
I=moment of inertia

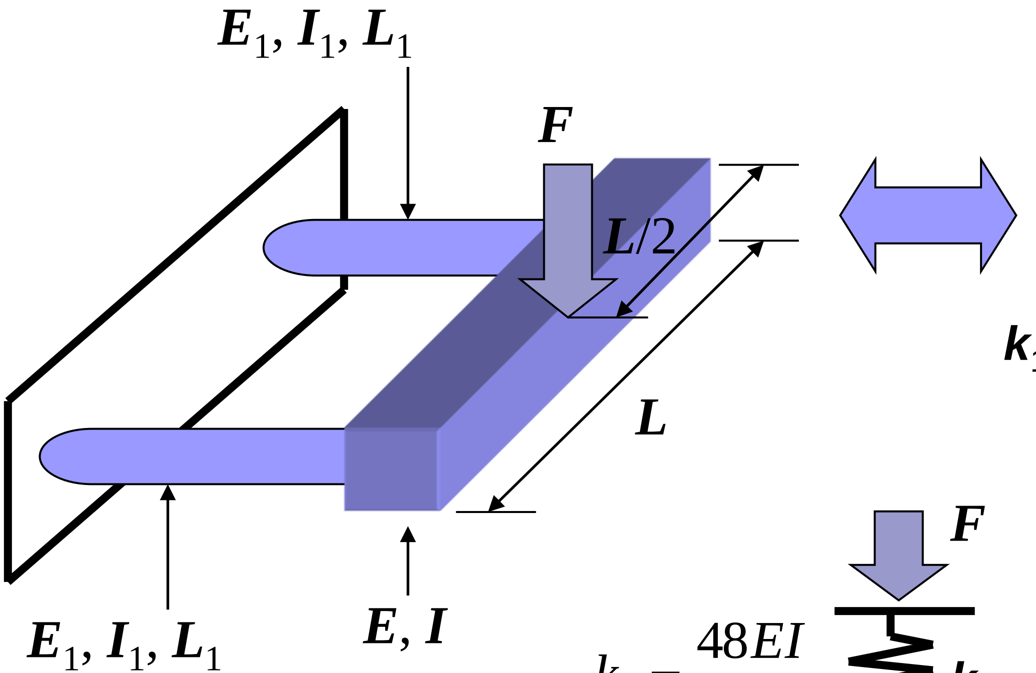


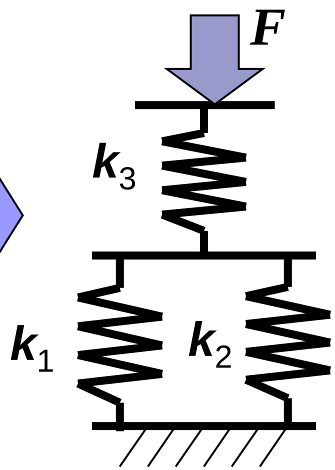


Cantilever beam: $$k_1 = k_2 = \frac{3E_1 I_1}{L_1^3}$$

Pinned-pinned beam: $$k_3 = \frac{48EI}{L^3}$$

Example 1



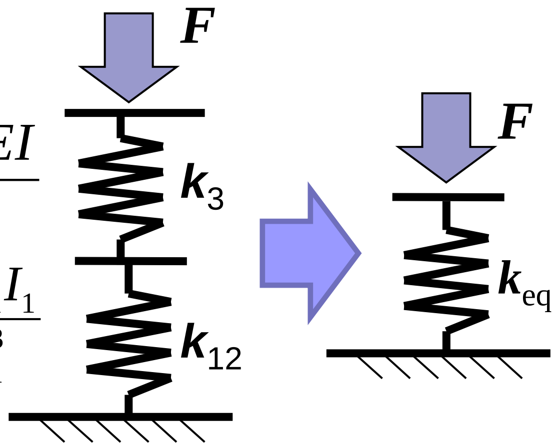


$$k_3 = \frac{48EI}{L^3}$$

$$k_1 = k_2 = \frac{3E_1I_1}{L_1^3}$$

$$k_3 = \frac{48EI}{L^3}$$

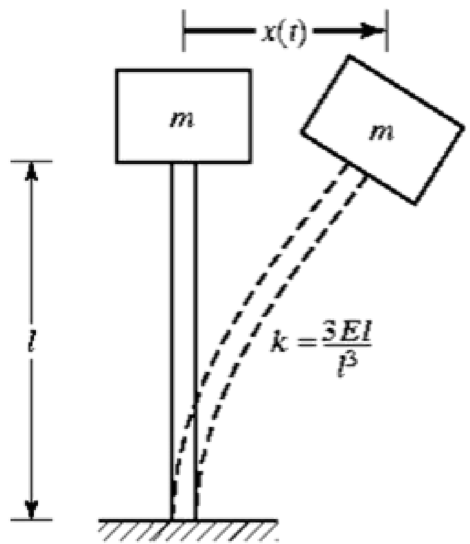
$$k_{12} = k_1 + k_2 = \frac{6E_1I_1}{L_1^3}$$



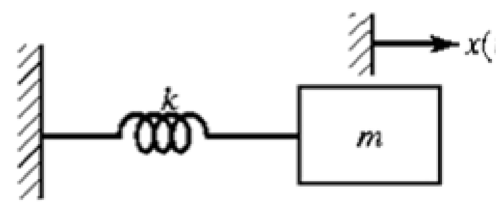
$$k_{eq} = \left(\frac{1}{k_{12}} + \frac{1}{k_3} \right)^{-1}$$

A simple spring-mass system

An example of a structure that can be idealized as simple spring-mass system if friction is negligible:

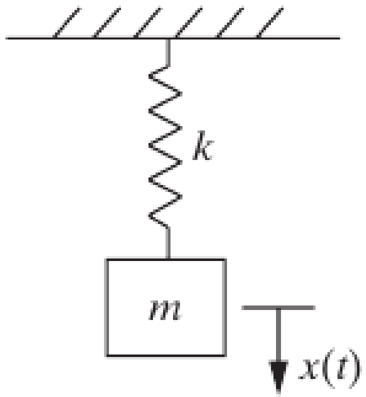


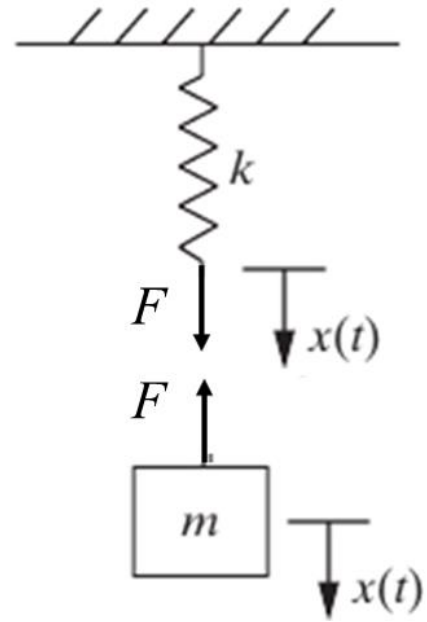
(a) Idealization of the tall structure

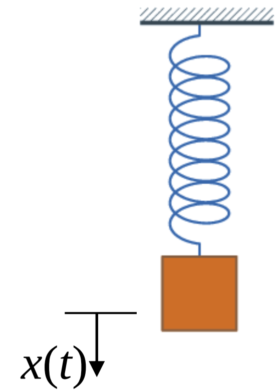


(b) Equivalent spring-mass system

A simple spring-mass system







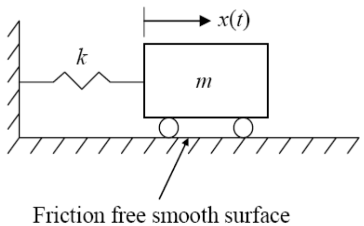
- At the equilibrium position, the spring extension balances the weight mg
- Draw the free-body diagram about the equilibrium position
- The spring force is $F = kx$
- Applying Newton's law on the mass

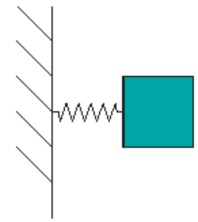
$$m\ddot{x} = -F = -kx$$

- The spring-mass equation is

$$m\ddot{x} + kx = 0$$

A simple spring-mass system





- A mass-spring system subjected to initial conditions x_0 and v_0 is an example of a single-degree of freedom "free vibration" undamped system
- The system equation has the form:

$$m\ddot{x} + kx = 0 \qquad \text{or} \qquad \ddot{x} + \omega_n^2 x = 0$$

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}}$
- The resulting response is simple harmonic motion

$$x(t) = A\sin(\omega_n t + \phi)$$

- Amplitude A
- Phase shift ϕ

Conservation of energy

A single-degree of freedom "free vibration" undamped system:

a) It is a conservative system as no energy is lost due to friction or energy-dissipating nonelastic members

b) No work is done on the conservative system by external forces (other than gravity or other potential forces)

❖ The total energy of the system remains constant:

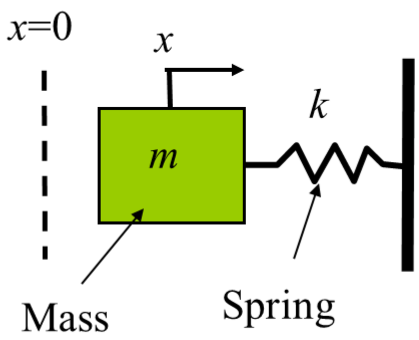
$$T + U = \text{constant}$$

- Kinetic energy $T = \frac{1}{2}m\dot{x}^2$ or $T = \frac{1}{2}I\dot{\theta}^2$
- Potential energy $U = \frac{1}{2}kx^2$ or $U = \frac{1}{2}k\theta^2$

❖ Total energy is constant also implies

$$\frac{d}{dt}(T + U) = 0$$

Conservation of energy



Consider the given spring-mass system:

- ❖ Kinetic energy $T = \frac{1}{2} m\dot{x}^2$
- ❖ Potential energy $U = \frac{1}{2} kx^2$
- ❖ Total energy of the system remains constant:

$$T + U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \text{constant}$$

- ❖ Total energy is constant also implies

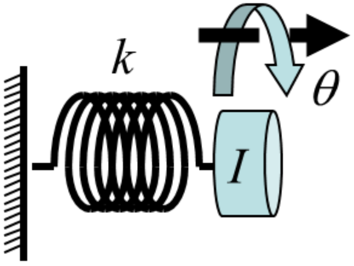
$$\frac{d}{dt}(T + U) = \frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2\right) = (m\dot{x}\ddot{x} + kx\dot{x}) = 0$$

$$(m\ddot{x} + kx)\dot{x} = 0$$

Since $\dot{x}$ cannot be zero for all time:

$$m\ddot{x} + kx = 0$$

Conservation of energy



Consider the equivalent spring-mass rotational system:

- Kinetic energy $T = \frac{1}{2} I\dot{\theta}^2$
- Potential energy $U = \frac{1}{2} k\theta^2$
- Total energy of the system remains constant:

$$T + U = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} k\theta^2 = \text{constant}$$

- Total energy is constant also implies

$$\frac{d}{dt}(T + U) = \frac{d}{dt}\left(\frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} k\theta^2\right) = \left(I\dot{\theta}\ddot{\theta} + k\theta\dot{\theta}\right) = 0$$

$$\left(I\ddot{\theta} + k\theta\right)\dot{\theta} = 0$$

Since $\dot{\theta}$ cannot be zero for all time:

$$I\ddot{\theta} + k\theta = 0$$

Lagrange's equation

Lagrange's equation is a method for deriving the equation of motion based on the energy

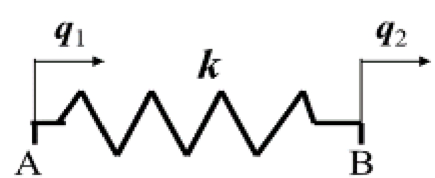
❖ In terms of generalized coordinate *q*, the Lagrange's equation for a single DOF free undamped system has the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0$$

- T = Kinetic energy
- U = Potential energy
- q = generalized coordinate that completely describe the dynamical system

Translational

Rotational



$$U = \frac{1}{2}k(q_1 - q_2)^2$$

Lagrange's Equations

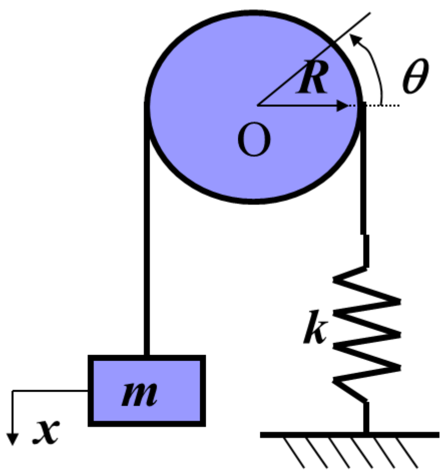
Procedure for 1DOF free undamped systems:

- Determine kinetic energy *T*,
- Find $\frac{\partial T}{\partial \dot{q}}$
- Find $\frac{\partial T}{\partial q}$
- Determine potential energy *U*,
- Find $\frac{\partial U}{\partial q}$
- Put the above into the equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0$$

Example 2

Use Lagrange's equation to derive the equation of motion for the system using generalized coordinate θ. Determine the system natural frequency. The mass moment of inertia of the disk about "O" is *I*.



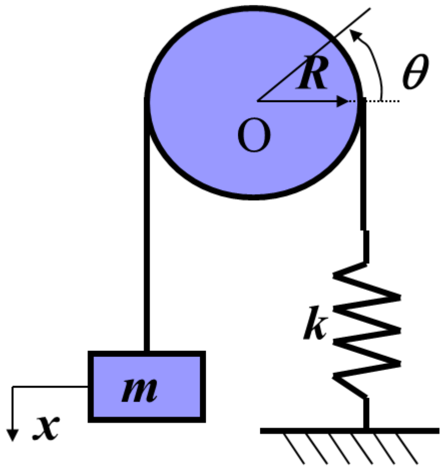
- Note that $x = R\theta$
- Generalized coordinate $q = \theta$
- Kinetic energy $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{\theta}} = I\dot{\theta} + mR^2\dot{\theta}$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial \theta} = 0$$

Example 2



- Potential energy $U = \frac{1}{2}kx^2 = \frac{1}{2}kR^2\theta^2$

$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial \theta} = kR^2\theta$$

- Apply Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = I\ddot{\theta} + mR^2\ddot{\theta} + kR^2\theta = 0$$

$$(I + mR^2)\ddot{\theta} + kR^2\theta = 0$$

- Natural frequency is

$$\omega_n = \sqrt{\frac{kR^2}{I + mR^2}}$$

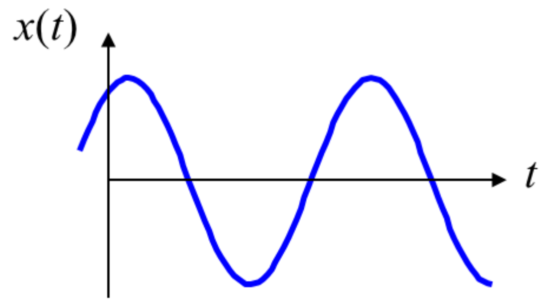
1DOF free undamped

A 1DOF free undamped system equation has the form:

$$m\ddot{x} + kx = 0$$

- Assumed a solution of the form:

$$x(t) = A\sin(\omega_n t + \phi)$$



- Differentiating twice gives:

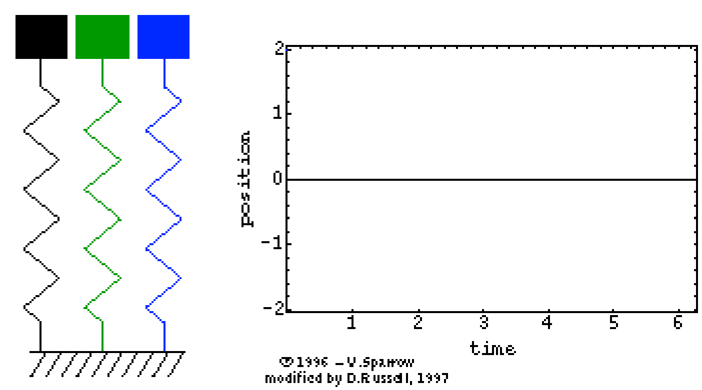
$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$$
$$\ddot{x}(t) = -\omega_n^2 A\sin(\omega_n t + \phi) = -\omega_n^2 x(t)$$

- Substituting these back into the system equation:

$$-m\omega_n^2 x(t) + kx(t) = 0$$

- Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$



- System equation can be written as:

$$\ddot{x} + \omega_n^2 x = 0$$

1DOF free undamped

- If the DOF free undamped system is vibrating then something must have (in the past) transferred energy into to the system and caused it to move
- For example the mass could have been moved a distance x_0 and then released at t = 0 (i.e. given Potential energy) or given an initial velocity v_0 (i.e. given Kinetic energy) or some combination of the two cases. These are called initial conditions
- Since $x(t) = A\sin(\omega_n t + \phi)$, initial displacement at time $t = 0$ is

$$x(0) = x_0 = A\sin(\phi)$$

- Similarly $\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi)$, initial velocity at time $t = 0$ is

$$\dot{x}(0) = v_0 = \omega_n A\cos(\phi)$$

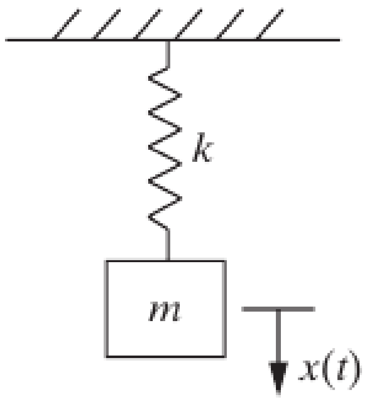
- Combining the 2 equations for the initial conditions:

$$\tan(\phi) = \frac{\omega_n x_0}{v_0} \text{ or } \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)$$

$$A^2(\sin(\phi))^2 + A^2(\cos(\phi))^2 = x_0^2 + (v_0/\omega_n)^2 \text{ or}$$

$$A = \sqrt{x_0^2 + (v_0/\omega_n)^2}$$

1DOF free undamped



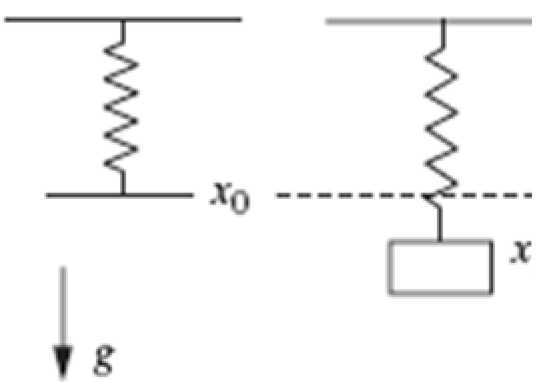
Spring-mass system:

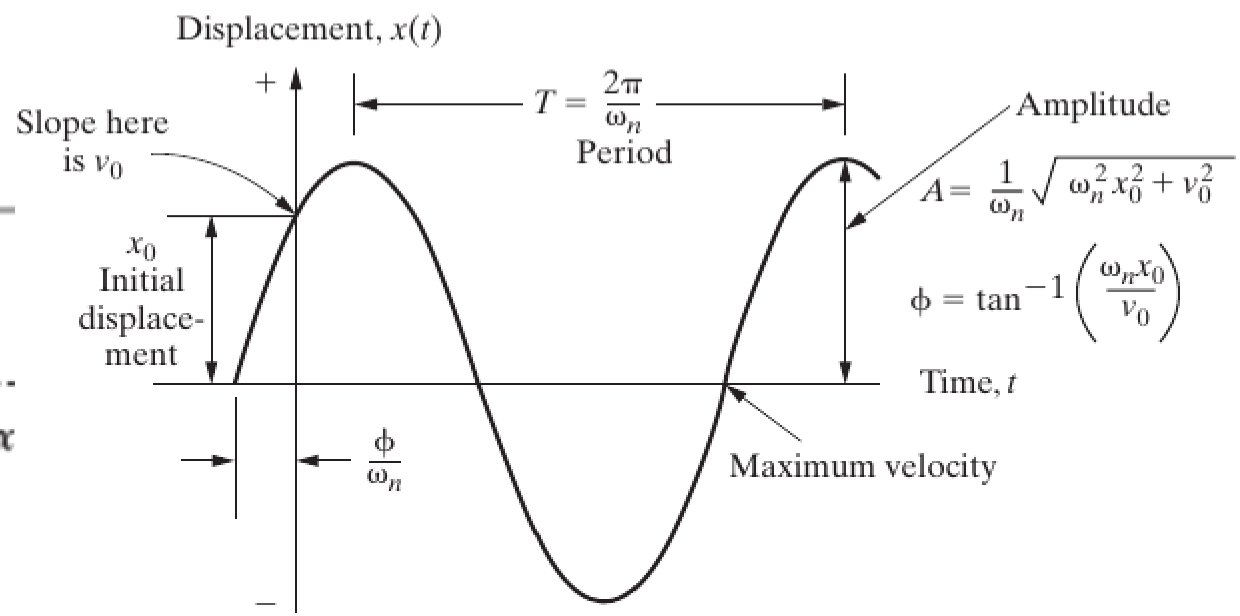
$$m\ddot{x} + kx = 0$$

With initial conditions x_0 and v_0

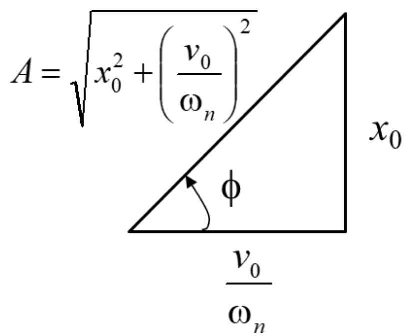
The resulting motion is periodic and has the form

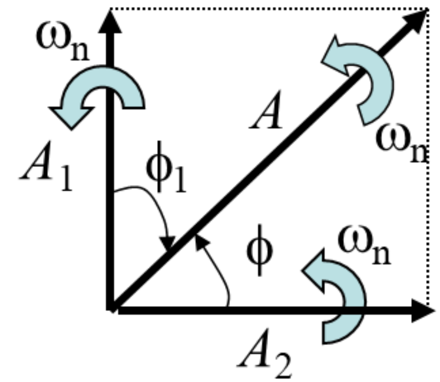
$$x(t) = A\sin(\omega_n t + \phi)$$





1DOF free undamped





Note: $\tan(\phi) = \frac{\omega_n x_0}{v_0} \Longrightarrow$

The solution to $m\ddot{x} + kx = 0$ is

$$x(t) = A\sin(\omega_n t + \phi)$$

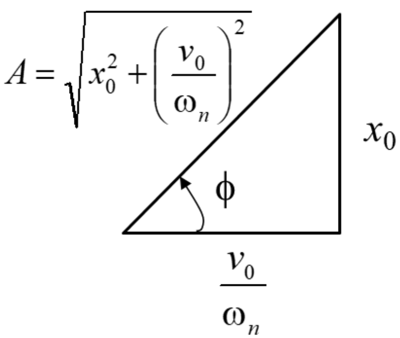
- This can be separated into 2 parts:

$$x(t) = A\sin(\omega_n t + \phi) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t)$$

or

$$x(t) = x_0\cos(\omega_n t) + \frac{v_0}{\omega_n}\sin(\omega_n t)$$

1DOF free undamped



Note:

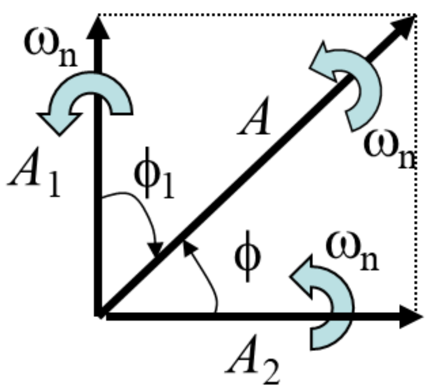
$$x(t) = A\sin(\omega_n t + \phi) = A_1\cos(\omega_n t) + A_2\sin(\omega_n t)$$

can also be written as

$$x(t) = A\cos(\omega_n t - \phi_1)$$

where

$$\phi_1 = \tan^{-1}\left(\frac{v_0}{\omega_n x_0}\right)$$



❖ The analysis also applies to 1 DOF free undamped rotational system by replacing

- initial conditions x_0 and v_0 with θ_0 and $\dot{\theta}_0$ respectively and
- mass with mass moment of inertia

1DOF free undamped

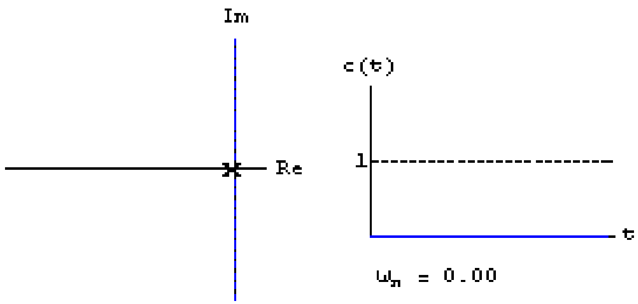
To solve $m\ddot{x} + kx = 0$ with initial conditions x_0 and v_0

We can also assume the solution has the form $x = Ae^{\lambda t}$

Substitute this back into the governing equation: $(m\lambda^2 + k)Ae^{\lambda t} = 0$

This is only satisfied for: $m\lambda^2 + k = 0$

$$\lambda_{1,2} = \sqrt{-\frac{k}{m}} = \pm\sqrt{\frac{k}{m}}j = \pm\omega_n j \quad \text{where } j = \sqrt{-1}$$



Solution is of the form:

$$x(t) = Ae^{\lambda t} = a_1 e^{j\omega_n t} + a_2 e^{-j\omega_n t}$$

$$x(t) = B\sin(\omega_n t + \phi)$$

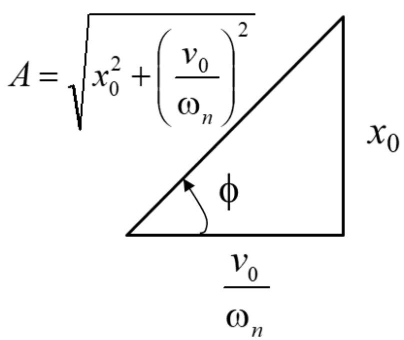
$$\dot{x}(t) = B\omega_n \cos(\omega_n t + \phi)$$

- At time $t = 0$, $x(0) = x_0$ and $\dot{x}(0) = v_0$:

$$x(0) = x_0 = B\sin(\phi)$$

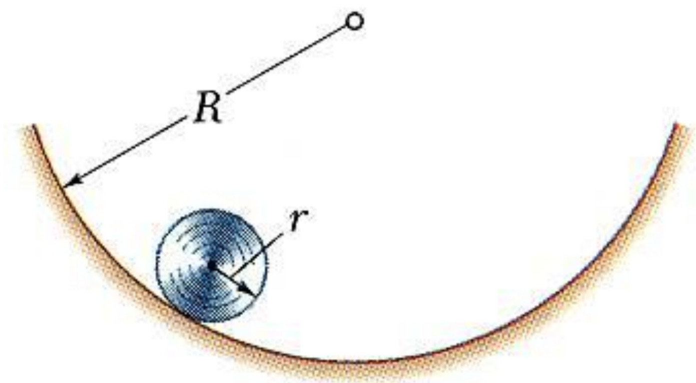
$$\dot{x}(0) = v_0 = B\omega_n \cos(\phi)$$

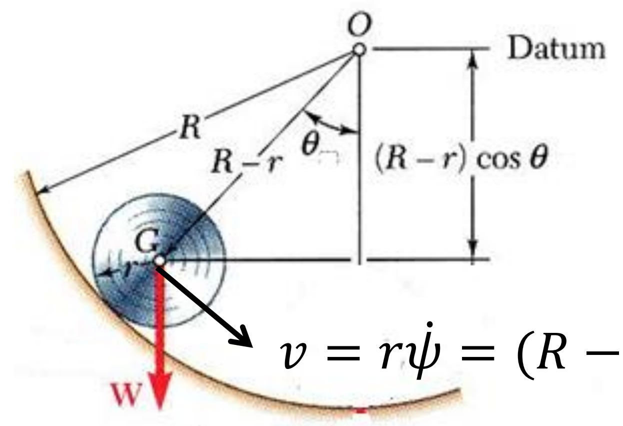
$$\tan(\phi) = \frac{\omega_n x_0}{v_0}$$



Example 3

Determine the system equation using Lagrange's equation and period of small oscillations of a cylinder which rolls without slipping inside a curved surface. If the system is displaced at rest by θ_0 at time $t = 0$, determine the response. Let Generalized coordinate $q = \theta$

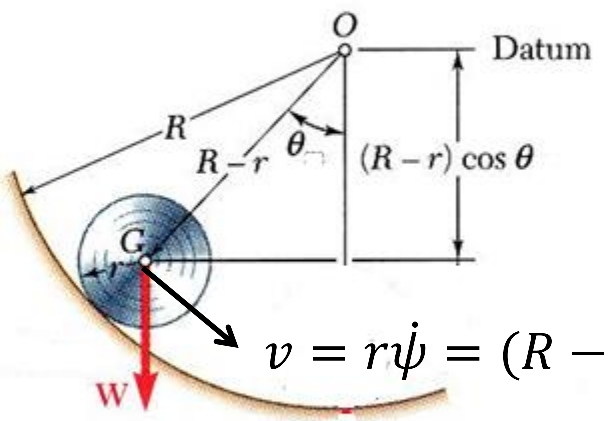




$$v = r\dot{\psi} = (R - r)\dot{\theta}$$

Note: Polar moment of inertia $I_G = \frac{1}{2}mr^2$

Example 3



$$v = r\dot{\psi} = (R-r)\dot{\theta}$$

- Note that $s = r\psi = (R-r)\theta$
  and $v = (R-r)\dot{\theta}$
- Generalized coordinate $q = \theta$
- Kinetic energy $T = \frac{1}{2} I_G \dot{\psi}^2 + \frac{1}{2} m v^2$

$$T = \frac{mr^2}{4}\left(\frac{R-r}{r}\right)^2 \dot{\theta}^2 + \frac{1}{2} m (R-r)^2 \dot{\theta}^2$$

$$\frac{\partial T}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{\theta}} = \frac{m}{2}(R-r)^2\dot{\theta} + m(R-r)^2\dot{\theta}$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial \theta} = 0$$

- Potential energy $U = -mg(R-r)\cos(\theta)$

$$\frac{\partial U}{\partial q} = \frac{\partial U}{\partial \theta} = mg(R-r)\sin(\theta)$$

Example 3

- Apply Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = 0$$

$$\frac{d}{dt}\left(\frac{m}{2}(R-r)^2 + m(R-r)^2\right)\dot{\theta} + mg(R-r)\sin(\theta) = 0$$

- For small θ: $\sin\theta \approx \theta$:

$$m(R-r)^2\left(\frac{1}{2}+1\right)\ddot{\theta} + m(R-r)g\theta = 0$$

$$(3/2)(R-r)\ddot{\theta} + g\theta = 0 \Longrightarrow \ddot{\theta} + \omega_n^2\theta = 0$$

- Natural frequency : $\omega_n = \sqrt{\frac{2g}{3(R-r)}}$

- Period: $\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$

Example 3

- Natural frequency : $\omega_n = \sqrt{\frac{2g}{3(R-r)}}$
- Period: $\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3(R-r)}{2g}}$
- Initial conditions at time $t = 0$: $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0 = 0$
- Amplitude $A = \sqrt{\theta_0^2 + \left(\dot{\theta}_0/\omega_n\right)^2} = \theta_0$
- Phase $\phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = \frac{\pi}{2}$
- System motion or response is:

$$x(t) = A\sin(\omega_n t + \phi) = \theta_0 \sin\left(\sqrt{\frac{2g}{3(R-r)}}t + \frac{\pi}{2}\right) = \theta_0 \cos\left(\sqrt{\frac{2g}{3(R-r)}}t\right)$$